From the Life of Units

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> "Alas!" said the Saturnian, "none of us live for more than five hundred annual revolutions of the sun; ... our existence is a point, our duration an instant, our globe an atom."
>
> Voltaire, *Micromegas*

Not so long agod it became known that certain seemingly quite different phenomena and processes that occur in life, for example, transformations of polymers, vital activity of biological tissues, crystal growth, and even the operation of computers in which calculations take place simultaneously in all the working cells (such a computer does not yet exist, but its construction would be a fascinating possibility) have nevertheless something in common. This common property is revealed when an attempt is made to describe the given processes in the language of mathematics and to create what researchers call a mathematical model.

The well-known Soviet physicist Ya. I. Frenkel′ said that a mathematical model should be more like a caricature than like a "realistic" picture. The main requirement of a model is simplicity; life processes are so complex that a realistic model yields to mathematical study with difficulty. Very widespread at present are so-called "lattice models" describing processes that take place on a lattice of identical "cells". In such models there are difficulties already when each cell has only two possible states. This is what we shall discuss below.

Colonies and operators

For a mathematician a piece of checked paper harbors many fascinating possibilities. We make use here of one of them. Open a page of a notebook of checked paper. You see on it a network of vertical and horizontal lines. For brevity we call the intersection points *nodes*. Let us imagine a world consisting solely of all these nodes. True, we do imagine the page here to be infinite. Suppose that some beings, called *units*, can live at these nodes. In figures we denote the units by circles as in Figure 1. Only one unit can be at a node at any one time, and if there is not a unit at a node, then there is a zero there. The collection of all nodes occupied by units will be called a *colony*. If there are finitely many units, then we say that the colony is *finite*. If there are no units at all, then we say that the colony is *empty*. (Like the space, the time is discrete, that is, it can be counted by integers: $0, 1, 2, \ldots$.)

The Russian original is published in *Kvant* 1974, no. 9, pp. 31–39.

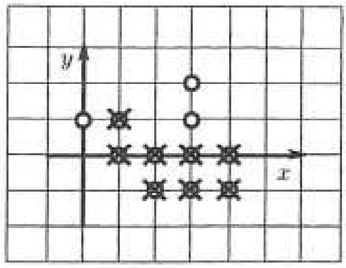

FIGURE 1

An *operator* $\mathbf{P}$ must be given, that is, a *rule* which, for any colony K existing at some time t, determines what colony $\mathbf{P}K$ is obtained from it at the next moment of time $t+1$.

Suppose that such an operator $\mathbf{P}$ is given. The basic question we are trying to answer is the following. *For the given operator* $\mathbf{P}$ *is there a finite colony that does not die out, that is, a finite colony* K *such that no matter how many times* $\mathbf{P}$ *is applied to it, the result will always be a nonempty colony? Or, on the contrary, is the operator* $\mathbf{P}$ *such that every finite colony dies out over some time interval, that is, not a single unit is left?*

Let us start by analyzing two examples. We introduce coordinate axes by taking one of the nodes as the origin O, the axis of abscissas to the right and the axis of ordinates upwards as usual (see Figure 1), and a side of a cell to be of length 1.

EXAMPLE 1. Suppose that at time $t+1$ there is a unit at the node $A=(x,y)$ if and only if *most of the following five nodes* were occupied by units at time t: A and its four neighbors, namely, $(x+1,y)$ to the right, $(x-1,y)$ to the left, $(x,y+1)$ above, and $(x,y-1)$ below. This defines the operator $\mathbf{P}$.

EXERCISE 1. What do the colonies in Figure 1 (the circles and the crosses) generate under the action of this operator? Follow the further evolution of the colonies. For this operator does there exist a finite colony that does not die out?

EXAMPLE 2 (see Problem M215). Suppose that at time $t+1$ there is a unit at the node $A=(x,y)$ if and only if *most of the following three nodes* were occupied by units at time t: A and its two neighbors $(x+1,y)$ and $(x,y+1)$ to the right and above. This defines the operator $\mathbf{P}$.

It follows from the solution of Problem M215 (published in *Kvant* 1974, no. 3) that under the action of this operator every finite colony dies out. (Verify this for the colony in Figure 1.) We shall prove this fact once more, by a method which will be useful to us in what follows.

Let us look at the evolution of colonies of a special form that fill isosceles right triangles with legs directed to the right and upwards from the vertex of the right angle (Figure 2). It turns out that each such triangle passes at the next moment of time into a triangle of the same form, but with legs shorter by one. The smallest triangle with legs of length 1 passes into a single point, and it disappears at the next step. (Verify this, by tracing the evolution of Figure 2.) Therefore, each such triangle dies out in a time equal to the original length of the legs.

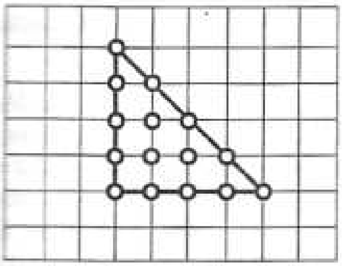

FIGURE 2

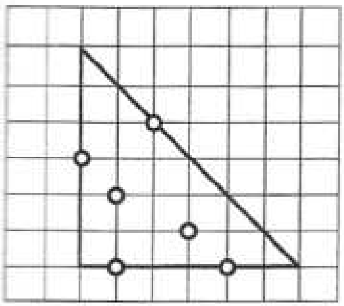

FIGURE 3

Now take an arbitrary finite colony K. It can be included in an isosceles right triangle, as follows from Figure 3. Then the colony obtained from it at the next moment will be inside a smaller triangle of the same form, and so on.

Here we have used an important property of the operator $\mathbf{P}$: if a colony K_1 is a part of a colony K_2 (this can be written as $K_1 \subset K_2$), then $\mathbf{P}K_1$ is a subset of $\mathbf{P}K_2$: $\mathbf{P}K_1 \subset \mathbf{P}K_2$ (the possibility that $\mathbf{P}K_1$ coincides with $\mathbf{P}K_2$ is not excluded). We call this property, which can be written briefly as

$$K_1 \subset K_2 \implies \mathbf{P}K_1 \subset \mathbf{P}K_2,$$

the *monotonicity* of the operator $\mathbf{P}$. Only *monotone* operators will be considered in this article.

For nonmonotone operators the question of when there are finite colonies that do not die out has not been solved in general form, and is possibly very difficult. Interesting and extensive material on the investigation of a particular nonmonotone operator, which has come to be called "The Game of Life", is contained in an article by M. Gardner in the October 1970 issue of *Scientific American* (p. 120).

We now describe the precise form of the operator $\mathbf{P}$ to be considered.

Let O be the "zero" node, the origin of coordinates. Part of the description of $\mathbf{P}$ is a *list* $\mathbf{U}$ consisting of the r nodes $u_1, \dots, u_r$ on whose states at time t the state of the node O at time $t+1$ depends. In Example 1 the number of these nodes is $r = 5$, and in Example 2 it is $r = 3$. The state of any node A at time $t+1$ depends on the states of the nodes $A + u_1, \dots, A + u_r$ at time t. The symbol $+$ here denotes addition of vectors. For example, if $A = (x, y)$ and $u_1 = (x', y')$, then $A + u_1 = (x + x', y + y')$.

EXERCISE 2. Write out the coordinates of the vectors $u_1, \ldots, u_5$ in Example 1 and the vectors u_1, u_2, u_3 in Example 2.

Besides the list $\mathbf{U}$, the specification of $\mathbf{P}$ includes the function determining just how the state of a node A at time $t+1$ depends on the states of the nodes $A+u_1, \ldots, A+u_r$ at time t. To specify this function we must indicate what will be at A (a unit or a zero) at time $t+1$ for *each* combination of units and zeros at the nodes $A+u_1, \ldots, A+u_r$ at time t. There are 2^r such combinations in all. For example, we can make a table in which there is either a unit or a zero opposite each combination of units and zeros, as done in Table 1 (verify that the function given by this table determines the operator $\mathbf{P}$ in Example 2). It is not necessary to make a table, of course. We can give the function verbally, though in such a way that such a table could be uniquely determined by the verbal description.

Functions $f = f(a_1, \ldots, a_r)$ whose arguments and values take only the two values 0 and 1 are called *Boolean* or *binary* functions. A Boolean function is said to be *monotone* if

$$f(a_1, \ldots, a_r) \leq f(a'_1, \ldots, a'_r)$$

whenever

$$a_1 \leq a'_1, \ldots, a_r \leq a'_r.$$

As already mentioned, we consider only monotone operators. They are given by monotone functions. Moreover, we set $f(0, \ldots, 0) = 0$ and $f(1, \ldots, 1) = 1$.

These restrictions are not essential, since the only monotone functions *not* satisfying them are the *constants*—the functions taking only one value (always zero or always one). The constant functions are very simple, and therefore uninteresting.

TABLE 1

States of nodes			
at time t			$t+1$
$(0,0)$	$(0,1)$	$(1,0)$	$(0,0)$
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

EXAMPLE 3. Suppose that at time $t+1$ there is a zero at the node $A = (x, y)$ if and only if at time t at least one of the following two conditions holds:

a) there are zeros at both the nodes (x, y) and $(x, y+1)$;

b) there are zeros at both the nodes $(x+1, y)$ and $(x+1, y+1)$.

This defines the operator $\mathbf{P}$.

EXERCISE 3. a) Trace the evolution of the colonies in Figure 1 under the action of this operator. Does one not get the impression that the colonies flatten out from the right side but stretch upward?

b) Prove that every finite colony dies out under the action of the operator in Example 3.

EXAMPLE 4. Suppose that at time $t+1$ there is a zero at the node $A = (x, y)$ if and only if at time t at least one of the following two conditions holds:

a) there are zeros at both the nodes (x, y) and $(x+1, y+1)$;

b) there are zeros at both the nodes $(x+1, y)$ and $(x, y+1)$.

This defines the operator **P**.

For this operator there do exist finite colonies that do not die out. It is true that they are constantly changing, but they never become extinct. Find one of them.

2. Zero sets. The survival theorem

We now remind ourselves that, besides the nodes, there are also other points in the plane. From the point of view of the units living only at nodes, of course, no other points exist in nature, but for us they do exist. We shall consider all possible geometric figures—sets of points in the plane. A figure **M** will be said to be *filled by zeros* if there are zeros at all the nodes contained in this figure.

DEFINITION 1. A figure **M** is called a zero figure (for a given operator **P**) if there will be a zero at the node O at time $t+1$ whenever **M** is filled by zeros at time t.

For instance, in Examples 1 or 2 a figure is a zero figure if it contains most of the five (in Example 1) or three (in Example 2) nodes making up the set **U**.

For a line l drawn in the plane we define a *half-plane* to be the set of points located on one side of the line, including the line itself.

Corresponding to Definition 1, a half-plane is called a *zero* half-plane if there will be a zero at the node O at time $t+1$ whenever it is filled by zeros at time t.

Let $\sigma_{\mathbf{P}}$ be the intersection of *all* the half-planes that are zero half-planes for the given operator **P**.

SURVIVAL THEOREM. *Every finite colony dies out under the action of the operator* **P** *if and only if the set* $\sigma_{\mathbf{P}}$ *is empty.*

EXERCISE 4 (Verification of the theorem for examples 1–4). Prove the following assertions.

a) In Example 2 the three half-planes shaded in Figure 4 are zero half-planes. In Example 3 the two half-planes shaded in Figure 5 are zero half-planes. In both cases the shaded half-planes do not have common points, that is, their intersection is empty. Therefore, the intersection $\sigma_{\mathbf{P}}$ of *all* the zero half-planes is empty. According to the theorem, all finite colonies die out.

b) In Example 1 the set $\sigma_{\mathbf{P}}$ consists of the single point O (and, according to the theorem, there is a finite colony that does not die out).

In Example 4 the set $\sigma_{\mathbf{P}}$ consists of the single point $\xi = (\frac{1}{2}, \frac{1}{2})$. (This is not a node, which is why nodes were not enough for us!)

In Example 1 the point O is in $\sigma_{\mathbf{P}}$, and a finite colony that does not die out remains in place. In Example 4 the simplest colony that does not die out consists of four nodes: the vertices of a unit square (Figure 6). At the next moment of time it generates a "cross", and this cross in turn generates the same square, but shifted

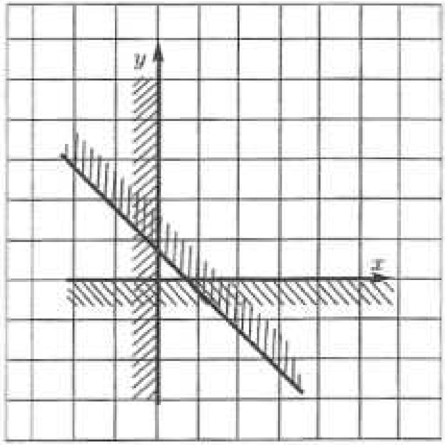

FIGURE 4

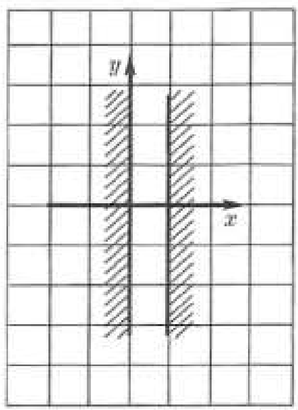

FIGURE 5

one unit to the left and downward. We say that a colony K *fills a figure* $\mathbf{M}$ if it contains all the nodes in $\mathbf{M}$. Both the cross and the square in Figure 6 can be represented as a colony filling identical figures—the squares shown in these figures. Thus, the colonies generated at successive moments of time can be represented as filling one and the same square, shifted by the vector $-\vec{\xi} = (-\frac{1}{2}, -\frac{1}{2})$ in unit time. Note that the sides of this square are parallel to the lines $u_i\xi$, where $u_i \in \mathbf{U}$. In Example 1 the sides of the unit square filled by the colony are also parallel to the lines u_iO.

EXERCISE 5 (teasing). If you were a unit and lived in the world ruled by the operator in Example 4, then would the point $\xi = (\frac{1}{2}, \frac{1}{2})$ exist for you or not?

3. Convex sets and convex hulls

A set $\mathbf{M}$ of points in the plane is said to be *convex* if it satisfies the following condition:

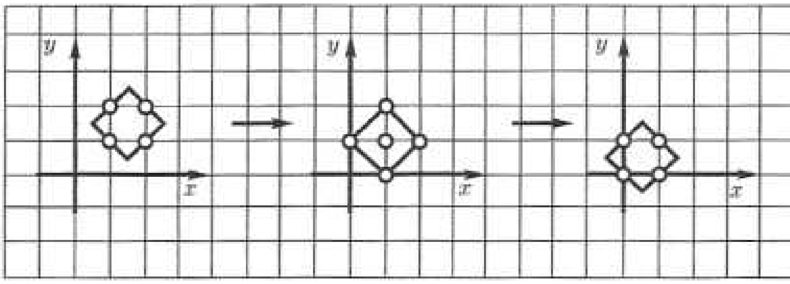

FIGURE 6

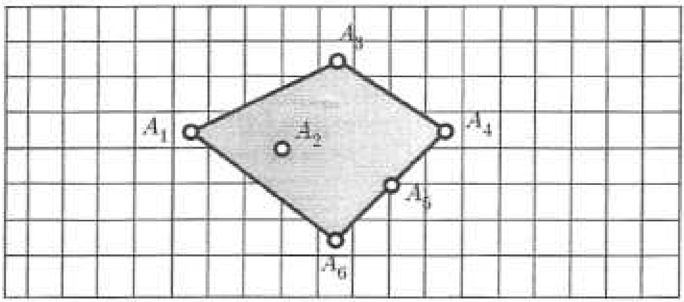

FIGURE 7

For any two points A and B in $\mathbf{M}$, *all the points of the straight line segment with endpoints A and B belong to* $\mathbf{M}$.

EXERCISE 6. Prove that every half-plane is a convex set.

EXERCISE 7. Prove that an intersection of convex sets is a convex set.

EXERCISE 8. In some textbooks a polygon is defined to be convex if it is located on one side of the line drawn through each of its sides. Prove that a polygon is convex in this sense if and only if the set of points located on or inside its contour is convex.

Thus, if a convex set $\mathbf{M}$ contains the points $A_1, \ldots, A_k$, then it contains also all the segments with endpoints at these points, and all the triangles with vertices at these points. The union of all the points $A_1, \ldots, A_k$, all the segments with endpoints at these points, and all the triangles with vertices at these points is called the *convex hull* of $A_1, \ldots, A_k$ (the convex hull of some points $A_1, \ldots, A_6$ is pictured in Figure 7). It can be said that the convex hull of a finite set of points is what is *necessarily* contained in every convex set containing the points.

EXERCISE 9. a) Prove that a set of four points (in the plane) can always be partitioned into two subsets whose convex hulls intersect.

b) The same statement when the set has $n > 4$ points.

Recall the definition of the set $\sigma_{\mathbf{P}}$. To find $\sigma_{\mathbf{P}}$ from this definition we must construct *all* zero half-planes, and there are infinitely many of them. True, in Examples 1–4 we knew how to find $\sigma_{\mathbf{P}}$, but it is not yet clear how to do this in the

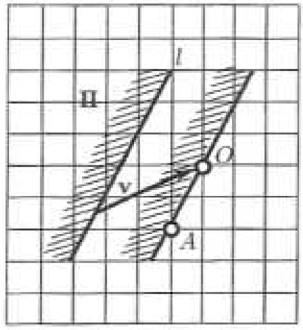

FIGURE 8

general case. It turns out that with the help of convex hulls it is now possible to construct $\sigma_{\mathbf{P}}$ for any operator $\mathbf{P}$.

We consider all subsets of $\mathbf{U}$. According to Definition 1, a set $\mathbf{U}' \subset \mathbf{U}$ is called a *zero* set if the fact that $\mathbf{U}'$ is filled by zeros at time t implies that there is a zero at O at the time $t+1$.

LEMMA 1. *$\sigma_{\mathbf{P}}$ coincides with the intersection of the convex hulls of all the zero subsets of $\mathbf{U}$.*

Prove this lemma yourself.

HINT. A half-plane is a zero half-plane if and only if it entirely contains at least one zero subset of $\mathbf{U}$.

Using Lemma 1, we can construct $\sigma_{\mathbf{P}}$ from any operator $\mathbf{P}$ given by its set $\mathbf{U}$ and function f. Namely, we must write out all the subsets of $\mathbf{U}$ (there are 2^r of them), then use our knowledge of f to choose the zero subsets among them, then construct their convex hulls (each is a polygon, segment, or point), and then find their intersection.

COROLLARY. *The set $\sigma_{\mathbf{P}}$ is always the intersection of a finite number of zero half-planes.*

EXERCISE 10. Using the method following from Lemma 1, construct again the set $\sigma_{\mathbf{P}}$ for Examples 1–4.

4. Proof of the survival theorem. The familiar Helly theorem in the role of a deus ex machina

A. Let us begin by finding conditions under which all finite colonies die out. First of all we make some important remarks. Suppose that a zero half-plane $\mathbf{\Pi}$ is bounded by a line l. In this case if $\mathbf{\Pi}$ was filled with zeros at time t, then at time $t+1$ there will be a zero not only at the node O, but also at every node A for which the line AO is parallel to l (Figure 8).

Further, we denote by $\mathbf{v}$ a vector by which l must be shifted in a parallel way in order to pass through O in its new position (there are infinitely many such vectors, but $\mathbf{v}$ can be taken to be any one of them). If at time t the zero half-plane $\mathbf{\Pi}$ is filled by zeros, then at time $t+1$ the half-plane $\mathbf{\Pi}+\mathbf{v}$ obtained from $\mathbf{\Pi}$ by shifting it by the vector $\mathbf{v}$ will be filled by zeros ($\mathbf{\Pi}+\mathbf{v}$ is shaded in Figure 8). Similarly, if

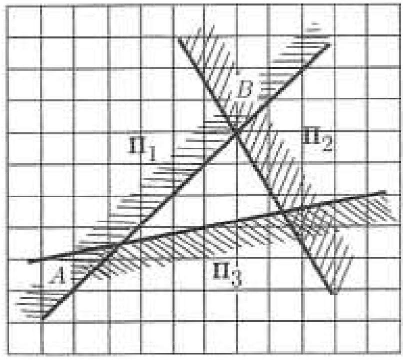

FIGURE 9

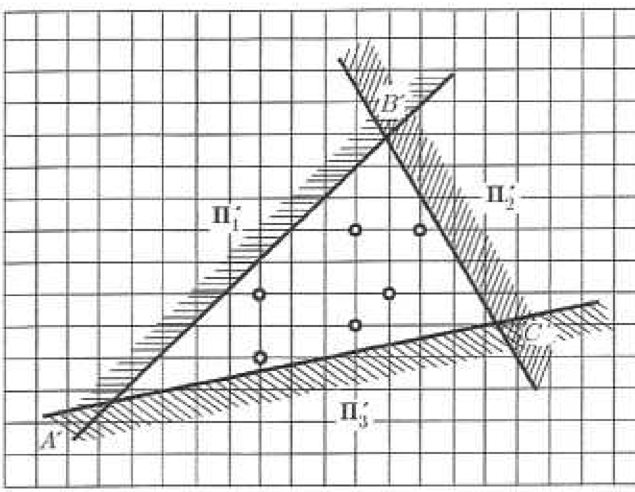

FIGURE 10

at time t the half-plane $\mathbf{\Pi} + \mathbf{w}$ is filled by zeros, then at time $t+1$ the half-plane $\mathbf{\Pi} + \mathbf{w} + \mathbf{v}$ is filled by zeros.

LEMMA 2. *Given a triangle ABC, assume that the three half-planes $\mathbf{\Pi}_1, \mathbf{\Pi}_2, \mathbf{\Pi}_3$ bounded by the lines AB, BC, AC and located outside ABC are zero half-planes for the operator* $\mathbf{P}$ *(they are shaded in Figure 9). Then every finite colony of units dies out under the action of this operator.*

PROOF. Let K be a finite colony. We shift the zero half-planes $\mathbf{\Pi}_1, \mathbf{\Pi}_2, \mathbf{\Pi}_3$ by vectors $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ such that the resulting half-planes $\mathbf{\Pi}_1' = \mathbf{\Pi}_1 + \mathbf{w}_1$, $\mathbf{\Pi}_2' = \mathbf{\Pi}_2 + \mathbf{w}_2$, $\mathbf{\Pi}_3' = \mathbf{\Pi}_3 + \mathbf{w}_3$ do not have points in common with K, that is, they are filled by zeros (Figure 10). The boundary lines of $\mathbf{\Pi}_1, \mathbf{\Pi}_2, \mathbf{\Pi}_3$ form a triangle $A'B'C'$ similar to ABC (and also oriented in the same way): denote the similarity coefficient by d. We prove that the colony K dies out in a time not greater than d.

Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ be vectors that shift the half-planes $\mathbf{\Pi}_1, \mathbf{\Pi}_2, \mathbf{\Pi}_3$ so that their boundary lines pass through O. From the similarity it then follows that if the half-planes $\mathbf{\Pi}_1', \mathbf{\Pi}_2', \mathbf{\Pi}_3'$ are shifted by the respective vectors $d\mathbf{v}_1, d\mathbf{v}_2, d\mathbf{v}_3$, then their boundary lines will also pass through a single point; these same half-planes in the new position cover the entire plane. But by our remarks, after t applications of $\mathbf{P}$ the half-planes obtained from $\mathbf{\Pi}'_1, \mathbf{\Pi}'_2, \mathbf{\Pi}'_3$ by shifting them by $t\mathbf{v}_1, t\mathbf{v}_2, t\mathbf{v}_3$ will certainly be filled by zeros, and for $t \geq d$ these half-planes cover the entire plane. Hence, the whole plane is filled by zeros, which means that the colony has died out.

LEMMA 3. *Suppose that two half-planes bounded by two parallel lines and disjoint from each other are zero half-planes for an operator* $\mathbf{P}$ *(as in Figure 5). Then every finite colony of units dies out under the action of this operator.*

Prove this lemma yourself.

HINT. Generalize the solution of Exercise 3.

Lemmas 2 and 3 can be combined into the following assertion.

LEMMA 4. *If the intersection of three zero half-planes is empty, then every finite colony of units dies out.*

PROOF. If the intersection of three half-planes is empty, then either they are arranged as in Figure 9, or some two of them are arranged as in Figure 5; but for these two cases the assertion of Lemma 4 is proved.

B. Suppose now that $\sigma_{\mathbf{P}}$ contains at least one point. We prove that there then exists a finite colony that does not die out.

LEMMA 5. *Suppose that an infinite colony* K *fills (at time* t*) a half-plane* $\mathbf{\Pi}$ *having at least one point in common with the set* $\sigma_{\mathbf{P}}$*. Then the colony* $\mathbf{P}K$ *contains the node* O.

PROOF. Assume that $\mathbf{P}K$ does not contain O. Then there is a zero at O at time $t+1$, that is, at time t some zero subset of $\mathbf{U}$ was entirely filled by zeros. Hence, this zero subset did not have any points in common with $\mathbf{\Pi}$, that is, it belonged to the *complement* of $\mathbf{\Pi}$. But the complement of $\mathbf{\Pi}$ is a convex set. Therefore, the convex hull of this zero subset was also entirely contained in the complement of $\mathbf{\Pi}$. Hence (see Lemma 1), $\sigma_{\mathbf{P}}$ was entirely contained in the complement of $\mathbf{\Pi}$, which contradicts our assumption.

COROLLARY. *Suppose that an infinite colony* K *fills a half-plane* $\mathbf{\Pi}$ *having at least one point in common with the set* $A+\sigma_{\mathbf{P}}$*, where* A *is a node. Then the colony* $\mathbf{P}K$ *contains* A.

The following statement is fundamental for the case B.

LEMMA 6. *Suppose that the point* ξ *belongs to* $\sigma_{\mathbf{P}}$ *(like the point* $(\frac{1}{2}, \frac{1}{2})$ *in Example 4). Then there is a convex polygon* $\mathbf{M}$ *such that if a colony* K *fills* $\mathbf{M}$*, then the colony* $\mathbf{P}K$ *fills the polygon* $\mathbf{M}-\vec{\xi}$ *(Figure 11), the colony* $\mathbf{P}^2K$ *fills* $\mathbf{M}-2\vec{\xi}$*, and so on; in general, the colony* $\mathbf{P}^nK$ *obtained from* K *in* n *units of time fills the polygon* $\mathbf{M}-n\vec{\xi}$ *obtained from* $\mathbf{M}$ *by a shift by the vector* $-n\vec{\xi}$.[1] *Furthermore, all the colonies* $K, \mathbf{P}K, \mathbf{P}^2K, \ldots$ *are nonempty.*

[1]We are asserting that all the nodes belonging to $\mathbf{M}-n\vec{\xi}$ are occupied by units, but possibly not only those nodes: it is not excluded that there are units also outside this polygon.

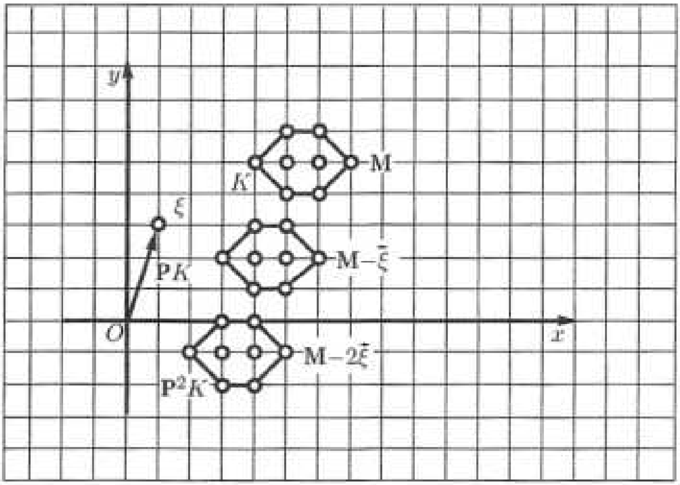

FIGURE 11

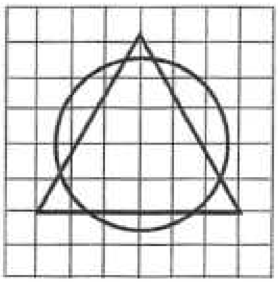

FIGURE 12

A particular consequence of this lemma is that if O belongs to $\sigma_{\mathbf{P}}$, then there is a "stable" colony that does not lose a single one of its nodes in the process of evolution (but that possibly acquires new nodes).

PROOF. Note that we are not claiming to have found the smallest colony that does not die out (as we did for Examples 1 and 4). It suffices for us to find any one such colony. Therefore, from the start we assume that $\mathbf{M}$ is very large. How large we say below.

It suffices to prove the assertion of the lemma for $n = 1$ alone; for larger n this will follow by induction. Accordingly, we assume that the colony K that exists at time t fills $\mathbf{M}$, and we prove that the colony $\mathbf{P}K$ fills the polygon $\mathbf{M} - \vec{\xi}$.

Suppose that some node A belongs to $\mathbf{M} - \vec{\xi}$ (see Figure 13). We show that A is in $\mathbf{P}K$ (this is written as $A \in \mathbf{P}K$). Obviously, $A + \xi \in \mathbf{M}$. Denote by R the largest of the distances from ξ to the nodes $u_1, \ldots, u_r$. Then the disk with center $A + \xi$ and radius R covers all the nodes $A + u_1, \ldots, A + u_r$. Choose $\mathbf{M}$ so large that every disk of radius R centered inside $\mathbf{M}$ does not cover points of the sides of $\mathbf{M}$ at all, or covers points only of one side of $\mathbf{M}$, or covers points only of two adjacent sides but no more. (Figure 12 shows a disk covering points of three sides of a triangle—this must not be!) Let us consider the three cases.

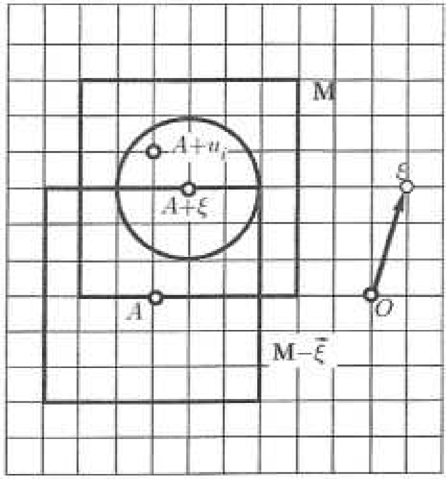

FIGURE 13

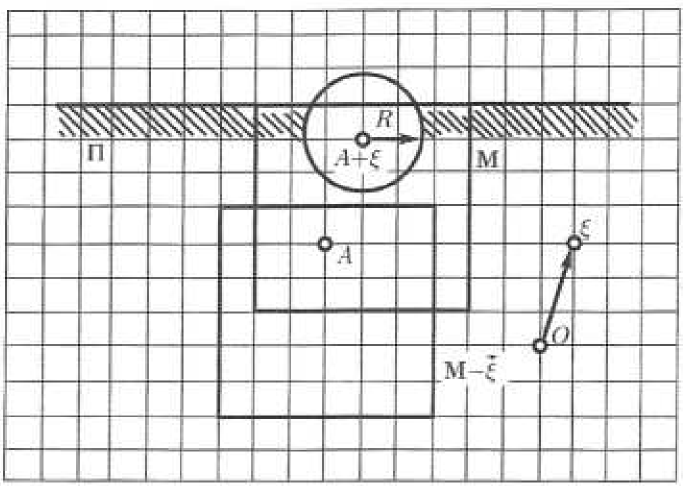

FIGURE 14

a) The disk centered at $A+\xi$ with radius R is entirely inside $\mathbf{M}$ (Figure 13). Then all the nodes $A+u_i$ are inside $\mathbf{M}$, that is, they are all in K; hence, the node A belongs to $\mathbf{P}K$.

b) The disk of radius R centered at $A+\xi$ covers points of only one of the sides of $\mathbf{M}$ (as in Figure 14). We draw the line l across this side. If the colony K filled the whole half-plane $\mathbf{\Pi}$ bounded by l (in Figure 14 this half-plane is shaded), then the colony $\mathbf{P}K$ would contain the node A by the corollary to Lemma 5 (the half-plane $\mathbf{\Pi}$ contains the point $A+\xi$, and hence intersects the set $A+\sigma_{\mathbf{P}}$). But the colony $\mathbf{P}K$ will contain A also when K fills only $\mathbf{M}$, because the state of A at time $t+1$ depends only on the states of the nodes $A+u_i$ at time t, and these nodes do not belong to the difference between $\mathbf{\Pi}$ and $\mathbf{M}$.

c) The disk of radius R with center $A+\xi$ covers points of two adjacent sides of $\mathbf{M}$ (Figure 15). We want to show that there is a unit at the node A at time $t+1$

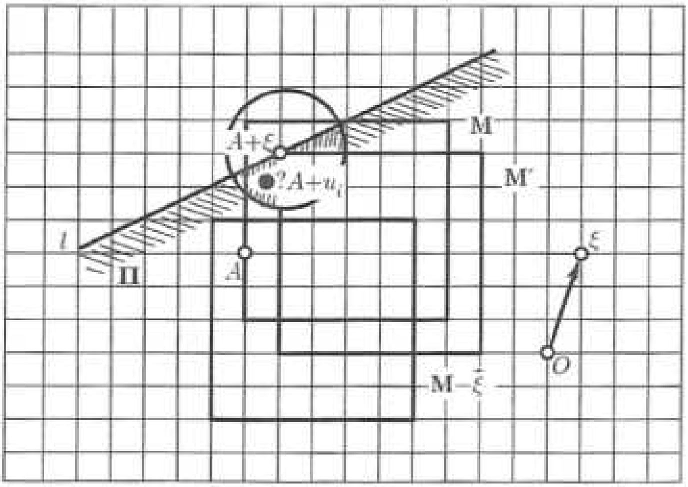

FIGURE 15

if at time t there are units at those points $A+u_1, \dots, A+u_r$ in the intersection of $\mathbf{M}$ with the disk. We prove that this actually holds even if there are units in a smaller area: the intersection of the same disk with the polygon $\mathbf{M}'$ obtained from $\mathbf{M}$ by a shift under which the common vertex of the two adjacent sides of $\mathbf{M}$ partly covered by the disk goes to the point $A+\xi$ (Figure 15). To this end we need to impose on $\mathbf{M}$ one more, now the last, condition.

Let us draw lines through the point ξ and each of the nodes $u_1, \dots, u_r$ different from ξ (if all the u_i are different from ξ, then there are r lines; if one of them coincides with ξ, then there are $r-1$ lines). We require that for each of these (r or $r-1$) lines there be two sides of $\mathbf{M}$ parallel to it. Then each line parallel to the line through ξ and some one of the points $u_1, \dots, u_r$ different from ξ either does not intersect $\mathbf{M}$ at all or has a whole segment in common with $\mathbf{M}$. For example, the square in Figure 6 (Example 4) satisfies this condition.

We now draw through the point $A+\xi$ a line l having only $A+\xi$ in common with the polygon $\mathbf{M}'$ (see Figure 15). If the colony filled the whole half-plane $\mathbf{\Pi}$ bounded by l (it is shaded in Figure 15), then by the corollary to Lemma 5, there would be a unit at A at time $t+1$. But the colony $\mathbf{P}K$ will contain the node A also when K fills only $\mathbf{M}$, since there are no nodes $A+u_i$ lying in $\mathbf{\Pi}$ but outside $\mathbf{M}'$. We prove this. Suppose that some node $A+u_i$ is in the difference between $\mathbf{\Pi}$ and $\mathbf{M}'$. But at the same time it is in the disk of radius R centered at $A+\xi$. Hence, the node $A+u_i$ lies in one of the two shaded sectors. Then the line passing through $A+u_i$ and $A+\xi$ has only one point in common with $\mathbf{M}'$, which is impossible in view of the choice of $\mathbf{M}$. This proves Lemma 6

C. *Helly's theorem.* Thus, we have investigated the following two cases:

A) the intersection of some three zero half-planes is empty (Lemma 4);

B) the intersection of all zero half-planes is nonempty (Lemma 6).

It turns out that only these cases are possible. This follows from the well-known theorem of Helly.

This is a very important and beautiful theorem, with many different formulations, proofs, and applications. We need it only in the following variant.

HELLY'S THEOREM. *Suppose that there are $n \geq 4$ convex sets in the plane, each three of which have a common point. Then all these n sets have a common point.*

Let us apply this to our case. By the corollary to Lemma 1. the set $\sigma_{\mathbf{P}}$ is the intersection of finitely many half-planes. Suppose that the case **B** does not hold: $\sigma_{\mathbf{P}}$ is *empty*. Then using Helly's theorem and arguing by contradiction, we get that the intersection of some *three* zero half-planes is empty; but this is the case **A**. This proves the survival theorem.