Teoria dos Números na Confluência da Álgebra, Análise e Combinatória.

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01. Given positive integers m and n, prove that is a positive integer c such that the numbers cm and cn have the same number of occurrences of each non-zero digit when written in base ten. (USAMO 2013)

02. For any $\epsilon > 0$ there exists an n_0 such that for all $n > n_0$ there are at least $\left(\frac{2}{3} - \epsilon\right)\frac{n}{log_2(n)}$ primes between n and $2n$. (Paul Erdös)

Minkowski's Theorem

Suppose that tha A is bounded centrally symmetric conver body in $\mathbb{R}^n$ having volume strictly greater than 2^n. Then there is a lattice point in A diferente from the origin.

03. Suppose that n is a positive integer for which the equation $x^2 + xy + y^2 = n$ has rational solutions. Then this equation has integer solutions as well. (Kömal)

Álgebra

04. Prove that for any integers $a_1, a_2, \dots, a_n$, the number

$$\prod_{1 \le i < j \le n} \frac{a_j - a_i}{j - i}$$

is an integer. (AMM – Armond Spencer)

05. Consider the sequence $(x_n)_{n\ge0}$ defined by $x_0 = 4,\ x_1 = x_2 = 0,\ x_3 = 3$ and $x_{n+4} = x_{n+1} + x_n$. Prove that for any prime p the number x_p is a multiple of p. (AMM)

06. Prove that the number

$$\sqrt{1001^2 + 1} + \sqrt{1002^2 + 1} + \cdots + \sqrt{2000^2 + 1}$$

is irracional. (China TST 2005)

07. Let $a_1, a_2, \dots, a_k$ be positive real numbers such that $\sqrt[n]{a_1} + \sqrt[n]{a_2} + \cdots + \sqrt[n]{a_k}$ is a rational number for all $n \ge 2$. Prove that $a_1 = a_2 = \cdots = a_k = 1$.

08. Let $f \in \mathbb{Z}[X]$ be a non-constant polynomial. Then the set of prime numbers dividing at least one nonzero number among

$$f(1), f(2), \dots, f(n), \dots$$

is infinite. (Schur)

09. Suppose that $f, g \in \mathbb{Z}[X]$ are monic nonconstant irreducible polynomials such that for all sufficiently large $n, f(n)$ and $g(n)$ have the same set of prime divisors. Then $f = g$.

10. Let $f \in \mathbb{Z}[X]$ be a nonconstant polynomial, and let $k \ge 2$ be na integer such that $\sqrt[k]{f(n)} \in \mathbb{Z}$ for all positive intergers n. Then there exists a polynomial $g \in \mathbb{Z}[X]$ such that $f = g^k$.

11. Let $f, g \in \mathbb{Z}[X]$ be two nonconstant polynomials such that $f(n)|g(n)$ for infinitely n. Prove that f divides g in $\mathbb{Q}[X]$.

12. Find all pairs of positive integers $m, n \ge 3$ for which there exist infinitely many positive integers a such that

$$\frac{a^m + a - 1}{a^n + a^2 - 1}$$

is itself na integer. (IMO 2002 – Laurentiu Panaitopol)

Análise

Kronecker's Theorem

The sequence $(\{na\})_{n\geq 1}$ is dense in $[0,1]$ if a is irracional.

13. Prove that the sequence $\left(\lfloor n\sqrt{2003}\rfloor\right)_{n\geq 1}$ contains arbitrarily long geometric progressions with arbitrarily large ratio. (Romanian TST 2003 – Radu Gologan)

14. Consider a positive integer *k* and a real number *a* such that log *a* is irrational. For each $n \geq 1$ let x_n be the number formed by the first *k* digits of $\lfloor a^n \rfloor$. Prove that the sequence $(x_n)_{n\geq 1}$ is not eventually periodical. (Mathlinks Contest – Gabriel Dopinescu)

15. Given a number $a \in \mathbb{N}^*$, prove that $\sigma(am) < \sigma(am+1)$ for infinitely many positive integers m. Here $\sigma(n)$ is the sum of all positive divisors of the positive integer number n. (tIMO Romania 2010 – Vlad Matei)

16. Define a positive integer n to be *squarish* if either n is itself a perfect square or the distance from n to the nearest perfect square is a perfect square. For, example, 2016 is squarish, because the nearest perfect square to 2016 is $45^2 = 2025$ and $2025 - 2016 = 9$ is a perfect square. (Of the positive integers between 1 and 10, only 6 and 7 are not squarish.)

For a positive integer N, let $S(N)$ be the number of squarish integers between 1 and N, inclusive. Find positive constants α and β such that

$$\lim_{N\to\infty} \frac{S(N)}{N^\alpha} = \beta,$$

or show that no such constants exist. (Putnam 2016)

Combinatória

Turán's Theorem

The greatest number of edges of a k-free graph with n vértices is

$$\frac{k-2}{k-1}\cdot\frac{n^2-r^2}{2}+\binom{r}{2},$$

where r is the remainder left by n when divided to $k-1$.

17. Let A be a subset of the set $S = \{1,2,\dots,1000000\}$ having exactly 101 elements. Prove that there exist $t_1, t_2, \dots, t_{100} \in S$ such that the sets $A_j = \{x + t_j \mid x \in A\}$ are pairwise disjoint. (IMO 2003 – Gugu)

18. Let n and k be nonnegative integers. Show that the number of partitions of n having k even parts is the same as the number of partitions of n in which the largest repeated part is k (defined to be 0 if the parts are all distinct). For example, 7 has three partitions with two even parts $(4+2+1 = 3+2+2 = 2+2+1+1+1)$ and also three partitions in which the largest repeated part is 2 $(3+2+2 = 2+2+2+1 = 2+2+1+1+1)$. (AMM – George E. Andrews and Emeric Deustch)

19. Let p be na odd prime number. How many p-element subsets A of $\{1,2,\dots,2p\}$ are there, the sum of whose elements is divisible by p? (IMO 1995 – Poland)

20. Find the number of ordered 64-tuples $(x_0, x_1, \dots, x_{63})$ such that $x_0, x_1, \dots, x_{63}$ are distinct elements of $\{1.2, \dots, 2017\}$ and

$$x_0 + x_1 + 2x_2 + 3x_3 + \cdots + 63x_{63}$$

Is divisible by 2017. (Putnam 2017)