IMC 2019, Blagoevgrad, Bulgaria

Day 1, July 30, 2019

Problem 1. Evaluate the product

$$\prod_{n=3}^{\infty} \frac{(n^3+3n)^2}{n^6-64}.$$

(10 points)

Problem 2. A four-digit number $YEAR$ is called *very good* if the system

$$\begin{aligned} Yx + Ey + Az + Rw &= Y \\ Rx + Yy + Ez + Aw &= E \\ Ax + Ry + Yz + Ew &= A \\ Ex + Ay + Rz + Yw &= R \end{aligned}$$

of linear equations in the variables x, y, z and w has at least two solutions. Find all very good YEARs in the 21st century.

(The 21st century starts in 2001 and ends in 2100.)

(10 points)

Problem 3. Let $f : (-1, 1) \to \mathbb{R}$ be a twice differentiable function such that

$$2f'(x) + xf''(x) \geq 1 \quad \text{for } x \in (-1, 1).$$

Prove that

$$\int_{-1}^{1} xf(x)\,\mathrm{d}x \geq \frac{1}{3}.$$

(10 points)

Problem 4. Define the sequence $a_0, a_1, \ldots$ of numbers by the following recurrence:

$$a_0 = 1, \quad a_1 = 2, \quad (n+3)a_{n+2} = (6n+9)a_{n+1} - na_n \quad \text{for } n \geq 0.$$

Prove that all terms of this sequence are integers.

(10 points)

Problem 5. Determine whether there exist an odd positive integer n and $n \times n$ matrices A and B with integer entries, that satisfy the following conditions:

1. $\det(B) = 1$;
2. $AB = BA$;
3. $A^4 + 4A^2B^2 + 16B^4 = 2019I$.

(Here I denotes the $n \times n$ identity matrix.)

(10 points)

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Day 2, July 31, 2019

Problem 6. Let $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ be continuous functions such that g is differentiable. Assume that $\big(f(0) - g'(0)\big)\big(g'(1) - f(1)\big) > 0$. Show that there exists a point $c \in (0, 1)$ such that $f(c) = g'(c)$.

(10 points)

Problem 7. Let $C = \{4, 6, 8, 9, 10, \ldots\}$ be the set of composite positive integers. For each $n \in C$ let a_n be the smallest positive integer k such that $k!$ is divisible by n. Determine whether the following series converges:

$$\sum_{n \in C} \left(\frac{a_n}{n}\right)^n.$$

(10 points)

Problem 8. Let $x_1, \ldots, x_n$ be real numbers. For any set $I \subset \{1, 2, \ldots, n\}$ let $s(I) = \sum\limits_{i \in I} x_i$. Assume that the function $I \mapsto s(I)$ takes on at least 1.8^n values where I runs over all 2^n subsets of $\{1, 2, \ldots, n\}$. Prove that the number of sets $I \subset \{1, 2, \ldots, n\}$ for which $s(I) = 2019$ does not exceed 1.7^n.

(10 points)

Problem 9. Determine all positive integers n for which there exist $n \times n$ real invertible matrices A and B that satisfy $AB - BA = B^2 A$.

(10 points)

Problem 10. 2019 points are chosen at random, independently, and distributed uniformly in the unit disc $\{(x, y) \in \mathbb{R}^2 \colon x^2 + y^2 \le 1\}$. Let C be the convex hull of the chosen points. Which probability is larger: that C is a polygon with three vertices, or a polygon with four vertices?

(10 points)