Combinatória Algébrica – Ed – SO 2020 – Natal RN

Estruturas Algébricas/Miscelânea

Probabilidade

1) Probabilistic Proof that $\mathbf{1+2+\cdots+n=\frac{n(n+1)}{2}}$.

Let X be the sum of two n-sided dice. For $k=2,3,\cdots,n+1$, the probability that $X=k$ is $(k-1)/n^2$ because there are $k-1$ ways of adding two positive integers to k, namely $1+(k-1), 2+(k-2),\dots,(k-1)+1$. For $k=n+i$ with $i=2,3,\cdots,n$, the probality that $X=k$ is $(n-i+1)/n^2$ because there are $n-i+1$ ways of adding up to k with two numbers from $\{1,2,\ \cdots,n\}$, namely $i+n,(i+1)+(n-1),\cdots,n+i$. Since $2\le X\le 2n$, then

$$1=\sum_{k=2}^{2n}\mathbb{P}[X=k]=\sum_{k=2}^{n+1}\frac{k-1}{n^2}+\sum_{i=2}^{n}\frac{n-i+1}{n^2}$$
$$1=\left(\frac{1}{n^2}+\frac{2}{n^2}+\cdots+\frac{n}{n^2}\right)+\left(\frac{n-1}{n^2}+\frac{n-2}{n^2}+\cdots+\frac{1}{n^2}\right)$$
$$n^2=(1+2+\cdots+n)+\big(1+2+\cdots+(n-1)\big)$$
$$n^2=2(1+2+\cdots+n)-n\cdot$$

Hence,

$$1+2+\cdots+n=\frac{n(n+1)}{2}\cdot$$

(Enrique Treviño, Lake Forest College)

Números Complexos

2) AMM 12032 [2018, 277] For a positive integer n, compute

$$\sum_{p=0}^{n}\sum_{k=p}^{n}(-1)^{k-p}\binom{k}{2p}\binom{n}{k}2^{n-k}.$$

Corpos Finitos

3) IMO-1993-6 There are n lamps $L_0,L_1,\cdots,L_{n-1}$ in a circle $(n>1)$, where we denote $L_{n+k}=L_k$. A lamp is on or off at all times. Perform steps $S_0,S_1,\cdots$ as follows: at step S_i, if L_{i-1} is lit, switch L_i from on to off or vice versa, otherwise do nothing. Initially all lamps are on. Show that:

a) There is a positive integer $\mathrm{M}(n)$ such that after $\mathrm{M}(n)$ steps all the lamps are on again;

b) If $n=2^k$, we can take $\mathrm{M}(n)=n^2-1$;

c) If $n=2^k+1$, we can take $\mathrm{M}(n)=n^2-n+1$.

Resolução Any state of the lamps $L_0,L_1,L_2,\cdots,L_{n-1}$ is determined by the vector $v=(v_1,v_1,\cdots,v_{n-1})$, where

$$v_i=\begin{cases}1, & \text{if lamp } L_i \text{ is on,}\\ 0, & \text{if lamp } L_i \text{ is off.}\end{cases}$$

The initial state is given by the vector $e=(1,1,\cdots,1)$. Note that

$$S_i\big((v_0,v_1,\cdots,v_{n-1})\big)=(v_0,v_1,\cdots,v_{i-1},v_{i-1}+v_i,v_{i+1},\cdots,v_{n-1}),$$

where + is the sum modulo 2, and hence $v_{i-1}+v_i\in\{0,1\}$.

a) It is sufficient to prove that $S_kS_{k-1}\cdots S_1S_0e=e$ for some $k\in\mathbb{N}$. Let T be the operation defined by $T\big((v_0,v_1,\cdots,v_{n-1})\big)=(v_1,\cdots,v_{n-1},v_0)$. Then, $S_i=T^{-i}S_0T^i$, and

$$S_kS_{k-1}\cdots S_1S_0=(T^{-k}S_0T^k)(T^{-k+1}S_0T^{k-1})\cdots(T^{-2}S_0T^2)(T^{-1}S_0T^1)S_0$$
$$=T^{-k}S_0(TS_0)^k=T^{-k-1}(TS_0)^{k+1}.$$

Note that the condition $S_kS_{k-1}\cdots S_1S_0e=e$ is equivalente to $(TS_0)^{k+1}e=e$. Let us consider the sequence $e,TS_0e,(TS_0)^2e,(TS_0)^3e,\dots$ Since there are 2^n possible states, it follows that there exist the positive integers i and j, such that $i > j$, and $(TS_0)^i e = (TS_0)^j e$. For $k = j - i - 1$ we obtain that $(TS_0)^{k+1} e = (TS_0)^{j-i} e = e$.

b) For $v = (v_0, v_1, \cdots, v_{n-1})$, let us define the polynomial

$$P_v(x) = v_{n-1}x^{n-1} + v_0 x^{n-2} + \cdots + v_{n-4}x^2 + v_{n-3}x + v_{n-2}.$$

Since $TS_0 v = T(v_{n-1} + v_0, v_1, \cdots, v_{n-1}) = (v_1, v_2, \cdots, v_{n-1}, v_{n-1} + v_0)$, it follows that

$$Q_v(x) := P_{TS_0 v}(x) = (v_{n-1} + v_0)x^{n-1} + v_1 x^{n-2} + \cdots + v_{n-3}x^2 + v_{n-2}x + v_{n-1}.$$

Note that $Q_v(x) \equiv xP_v(x) (\mathrm{mod}\ x^n - x^{n-1} - 1)$. It follows that the equality $(TS_0)^k e = e$ is equivalent to $x^k \equiv 1\ (\mathrm{mod}\ x^n - x^{n-1} - 1)$. For $n = 2^k$ we obtain that

$$x^{n^2} \equiv (x^n)^n \equiv (x^{n-1} + 1)^n \equiv x^{n^2-n} + 1, \qquad (1)$$

where $\equiv$ is always congruence modulo $x^n - x^{n-1} - 1$. Equality (1) holds because all the binomial coefficients $\binom{2^k}{1}, \binom{2^k}{2}, \cdots, \binom{2^k}{2^k-1}$ are even positive integers, i.e., equal to 0 modulo 2. It follows from (1) that

$$1 \equiv x^{n^2} - x^{n^2-n} \equiv x^{n^2-n}(x^n - 1) \equiv x^{n^2-n} \cdot x^{n-1} \equiv x^{n^2-1}.$$

Hence, after $n^2 - 1$ steps all the lamps will be on again.

c) Let $n = 2^k + 1$. Then similarly as in the previous case we obtain that

$$x^{n^2-1} \equiv (x^{n+1})^{n-1} \equiv [x(x^{n-1} + 1)]^{n-1} \equiv (x^n + x)^{n-1} \equiv x^{n(n-1)} + x^{n-1},$$

It follows that $x^{n^2-1} - x^{n(n-1)} \equiv x^{n-1}$, i.e., $x^{n^2-n}(x^{n-1} - 1) \equiv x^{n-1}$. Hence $x^{n^2-n} \cdot x^n \equiv x^{n-1}$, and finally $x^{n^2-n+1} \equiv 1$. The conclusion is that after $n^2 - n + 1$ steps all the lamps will be on again.

Burnside's Lemma

4) Honsberger's Theorem For any positive integer n, the number of incongruent triangles with integer sides and perimeter n is $\left[\frac{n^2}{48}\right]$ if n is even, or $\left[\frac{(n+3)^2}{48}\right]$ if n is odd.

Polinômios Simétricos

5) MOSER POLYNOMIALS/Dmitri V. Fomin, AMM May 2019

Notation. For any pair of positive integers n and s such that $n \geq s$, let $A^{(s)}$ denote the multiset of s-sums of (set) A, i.e., the colllection of all sums of the form

$$a_{i_1} + a_{i_2} + \cdots + a_{i_s},$$

Where $1 \leq i_1 < i_2 < \cdots < i_s \leq n$.

Given an n-multiset $A = \{a_1, \cdots, a_n\}$, we can produce a sequence of sums of its kth powers for $k = 1, \cdots, n$. That is, we can apply power-sum symmetric polynomials in n variables,

$$\sigma_k(x_1, \cdots, x_n) = \sum_{i=1}^{n} x_i^k,$$

to multiset A to obtain the sequence $\sigma_1(A), \cdots, \sigma_n(A)$. It is well known that A can be restored from this sequence – the values of $\sigma_k(A)$ determine coefficients of the polynomial $(x - a_1)(x - a_2) \cdots (x - a_n)$ and therefore determine the multiset of its roots.

Thus if all values of $\sigma_k(A)$ for $k = 1, \cdots, n$ can be deduced from values of $\sigma_k(A^{(s)})$, then multiset A is uniquely determined by multiset $A^{(s)}$.

Let us start with small values of k. For $k = 1$ we have

$$\sigma_1(A^{(s)}) = \binom{n-1}{s-1} \sigma_1(A)$$

and therefore, if $A^{(s)} = B^{(s)}$, then $\sigma_1(A) = \sigma_1(B)$. This means that $\sigma_1(A)$ can always be found from $A^{(s)}$.

Now, if $k = 2$, then

$$\sigma_2(A^{(s)}) = \sum_{1 \leq i_1 < \cdots < i_s \leq n} (a_{i_1} + \cdots + a_{i_s})^2$$

$$= \binom{n-1}{s-1} \sum_{1 \leq i \leq n} a_i^2 + \binom{n-2}{s-2} \sum_{1 \leq i < j \leq n} 2a_i a_j$$

$$= \binom{n-1}{s-1} \sum_{1 \leq i \leq n} a_i^2 + \binom{n-2}{s-2} \sum_{1 \leq i \neq j \leq n} a_i a_j$$

$$= \left(\binom{n-1}{s-1} - \binom{n-2}{s-2}\right) \sum_{1\le i\le n} a_i^2 + \binom{n-2}{s-2} \sum_{1\le i,j\le n} a_i a_j$$
$$= \binom{n-2}{s-1}\sigma_2(A) + \binom{n-2}{s-2}\sigma_1(A)^2 \cdot$$

We already know $\sigma_1(A)$ and thus we can find $\sigma_2(A)$ as long as the coefficient $\binom{n-2}{s-1}$ is not zero. For $n > s > 0$ that is always true and therefore $\sigma_2(A)$ is also always "recoverable".
Since $\sigma_k(A^{(s)})$ is a symmetric polynomial in $a_1, \cdots, a_n$, it can be expressed in the following way:

$$\sigma_k(A^{(s)}) = \alpha_{s,k,n}\sigma_k(A) + \mathcal{P}(\sigma_1(A), \sigma_2(A), \cdots, \sigma_{k-1}(A)) \ (2)$$

where $\alpha_{s,k,n}$ is a constant (in terms of the variables a_i) defined by the three numbers s, k, n, and $\mathcal{P}$ is some polynomial in the $k-1$ variables $\sigma_1, \cdots, \sigma_{k-1}$, whose coefficients are fully defined by the triplet s, k, n as well. For instance, as we have just shown, for $k = 2$ we have $\alpha_{s,k,n} = \binom{n-2}{s-1}$ and $\mathcal{P}(\sigma) = \binom{n-2}{s-2}\sigma^2$.
It follows that if we could show that the coefficient $\alpha_{s,k,n}$ does not vanish, then we will have proved that $\sigma_k(A)$ is determined by $\sigma_1(A), \sigma_2(A), \cdots, \sigma_{k-1}(A)$, $\sigma_k(A^{(s)})$. Thus, if $\alpha_{s,k,n} \neq 0$ for all $1 \le k \le n$, then using this for $k = 1$, then for $k = 2$, etc., we can conclude that multiset A can be recovered from the multiset $A^{(s)}$ and the pair (n, s) is not *singular*.
The following equality holds.

Theorem

$$\alpha_{s,k,n} = \sum_{p=1}^{s} (-1)^{p-1} p^{k-1} \binom{n}{s-p} \cdot \ (3)$$

So the coefficient $\alpha_{s,k,n}$turns out to be a polynomial in n.
Definition. We will call the right-hand side of equation (3) the **Moser polynomial** and will denote it by $F_{s,k}(n)$. That is,

$$F_{s,k}(n) = \sum_{p=1}^{s} (-1)^{p-1} p^{k-1} \binom{n}{s-p} = \sum_{j=0}^{s} (-1)^{s-j-1} (s-j)^{k-1} \binom{n}{j} \cdot$$

We will also set $F_{s,k}(n) = 0$ for any nonpositive integer s. In this way Moser polynomials are defined for any integer s and natural number k. The **normalized Moser polynomial** $(s-1)!\,F_{s,k}$ has integer coefficients and will be denoted by $\tilde{F}_{s,k}$.
We will leave it to the reader as an exercise to prove that for $k = 2$ formula (3) is indeed equivalent to $F_{s,2}(n) = \binom{n-2}{s-1}$.
Case $s = 2$. Incidentally, even without formula (3) the case $s = 2$ can now be resolved in a very straightforward manner. Computing $\sigma_k(A^{(2)})$ we obtain (using notation from (2))

$$\alpha_{2,k,n} = n - 2^{k-1},$$

which means that the n-multiset is always recoverable from the multiset of its 2-sums if n is not a power of 2. As we already know, if n is a power of 2, then the n-multiset A cannot always be recovered from $A^{(2)}$.
Case $s = 3$. Equation (3) gives us

$$F_{3,k}(n) = \binom{n}{2} - 2^{k-1}\binom{n}{1} + 3^{k-1},$$
$$\tilde{F}_{3,k}(n) = n^2 - n(2^k + 1) + 2 \cdot 3^{k-1} \cdot$$

Investigation here is again relatively straightforward. First, we can prove that $F_{3,k}(n)$ cannot be zero for positive n if $k > 12$. Second, we check all the cases with $k \le 12$ and verify that the polynomials $F_{3,}k$ (or $\tilde{F}_{3,k}$) have integer roots if and only if $k \in \{1, 2, 3, 5, 9\}$. For these five especial cases we have

$$\tilde{F}_{3,1}(n) = (n-1)(n-2),$$
$$\tilde{F}_{3,2}(n) = (n-2)(n-3)$$
$$\tilde{F}_{3,3}(n) = (n-3)(n-6),$$
$$\tilde{F}_{3,5}(n) = (n-6)(n-27),$$

$$\tilde{F}_{3,9}(n) = (n-27)-(n-486).$$

The pair $(6,3)$ is singular (verify!). To prove the same for the pairs $(27,3)$ and $(486,3)$ we present the following examples:

$$n = 27: \quad A_{27} = \{0, 1^{16}, 2^{10}\}, \{0^5, 1^{10}, 2^{10}, 3^2\}, \{0, 1^5, 2^{10}, 3^6, 4^5\}.$$

$$n = 486: \qquad A_{486} = \{0^{22}, 1^{176}, 2^{231}, 3^{56}, 4\}.$$

We will leave it to the reader to verify that each one of these four multisets is 3-equivalent to its mirror. Nowadays, this can be done in minutes, using just a few lines of code in some decent computation package.

Case $s = 4$. From (3) we obtain

$$F_{4,k}(n) = \binom{n}{3} - 2^{k-1}\binom{n}{2} + 3^{k-1}\binom{n}{1} - 4^{k-1},$$

$$\tilde{F}_{4,k}(n) = n^3 - n^2(3 + 3 \cdot 2^{k-1} + 1) + n(2 + 3 \cdot 2^{k-1} + 2 \cdot 3^k) - 6 \cdot 4^{k-1}.$$

Using divisibility and other relatively straightforward ideas from elementary number theory we can prove that for $k > 7$ the polynomials $F_{4,k}$ do not have positive integer roots. And, finally,

$$\tilde{F}_{4,1}(n) = (n-1)(n-2)(n-3),$$

$$\tilde{F}_{4,2}(n) = (n-2)(n-3)(n-4),$$

$$\tilde{F}_{4,3}(n) = (n-3)(n-4)(n-8),$$

$$\tilde{F}_{4,4}(n) = (n-4)(n^2 - 23n + 96),$$

$$\tilde{F}_{4,5}(n) = (n-8)(n^2 - 43n + 192),$$

$$\tilde{F}_{4,6}(n) = (n-12)(n^2 - 87n + 512),$$

$$\tilde{F}_{4,7}(n) = (n-8)(n^2 - 187n + 3072).$$

The only nontrivial root of the $F_{4,k}$ polynomials, other than $n = 2s = 8$, is 12. As we mentioned before, this case turned out to be tougher than the others—a pair of 4-equivalent 12-multisets was found only 2016 by Isomurodov and Kokhas. Namely, if we consider the following two different 12-multisets

$$A = \{1^2, 4, 6, 7, 8^2, 9, 10, 12, 15^2\}, B = \{0, 3, 4, 5, 6, 7, 9, 10, 11, 12, 13, 16\},$$

the authors actually proved that this is the only possible singularity example for case $(12,4)$ (where we consider pairs of multisets that differ only by linear transformation to be identical).

Funções Geratrizes (Contagem/Partições)

6) Base-b Representations

For $b > 1$, is the existence of base-b representations for positive integers is typically proved using either strong induction or the division algorithm, while uniqueness of the representation is typically obtained via a proof by contradiction. However, existence and uniqueness can be simultaneously obtainded through a short generating function argument.

One needs to show that if n, b and k are positive integers such that $b > 1$ and $0 \le n < b^{k+1}$, then there exist unique integers $x_0, x_1, \cdots, x_k$ with $0 \le x_i < b$ such that

$$n = x_0 + x_1 b + x_2 b^2 + \cdots + x_k b^k = \sum_{i=0}^{k} x_i b^i \ .$$

Observe that each expression $x_0 + x_1 b + x_2 b^2 + \cdots + x_k b^k$ with $0 \le x_i < b$ for all i is an integer partition using parts from $\{1, b, b^2, \cdots, b^k\}$ where each part appears less than b times. The generating function for these partitions is

$$\sum_{0 \le x_0, x_1, \cdots, x_k < b} z^{x_0 + x_1 b + x_2 b^2 + \cdots + x_k b^k}$$

$$= \prod_{i=0}^{k} \left(1 + z^{b^i} + z^{2.b^i} + z^{3.b^i} + \cdots + z^{(b-i).b^i}\right)$$

$$= \frac{\prod_{i=0}^{k}\left(1 - z^{b.b^i}\right)}{\prod_{i=0}^{k}\left(1 - z^{b^i}\right)}$$

$$= \frac{1 - z^{b.b^k}}{1 - z} = 1 + z + z^2 + \cdots + z^{b^{k+1}-1} .$$

Thus, there is exactly one such partition for each integer between 0 and $b^{k+1} - 1$.

7) Putnam 2018 B6 Let S be the set of sequences of lenght 2018 whose terms are in the set $\{1, 2, 3, 4, 5, 6, 10\}$ and sum to 3860. Prove that the cardinality of S is at most

$$2^{3860} \cdot \left(\frac{2018}{2048}\right)^{2018} .$$

Resolução (Murilo Corato Zanarella, Princeton University) Consider the polynomial

$$P(x) = (x + x^2 + x^3 + x^4 + x^5 + x^6 + x^{10})^{2018}.$$

The cardinality $|S|$ of S is the coefficient of x^{3860} in $P(x)$. Since

$$\frac{2018}{2048} = \frac{2^{10} + 2^9 + 2^8 + 2^7 + 2^6 + 2^5 + 2}{2^{11}}$$
$$= 2^{-1} + 2^{-2} + 2^{-3} + 2^{-4} + 2^{-5} + 2^{-6} + 2^{-10},$$

we have $P(1/2) = (2018/2048)^{2018}$. Because all coefficients of $P(x)$ are nonnegative, the value $P(1/2)$ is at least the value of the term $|S|x^{3860}$ for $x = 1/2$. Therefore,

$$\left(\frac{2018}{2048}\right)^{2018} \geq |S| \left(\frac{1}{2}\right)^{3860},$$

which gives the desired inequality

$$|S| \leq 2^{3860} \cdot \left(\frac{2018}{2048}\right)^{2018} .$$

8) AMM 11985 [2017,563] For fixed $s, t \in \mathbb{N}$ with $s \leq t$, let $a_n = \binom{n}{s} + \binom{n}{s+1} + \cdots + \binom{n}{t}$. Prove that this sequence is log-concave, namely that $a_n^2 \geq a_{n-1}a_{n+1}$ for $n \geq 1$.

9) THE GASARCH-KRUSKAL THEOREM/Ian Morrison, AMM August-September 2019 We define notions and notation used in the sequel and then state and prove the Gasarch-Kruskal theorem streamlining the original arguments slightly.

A die d of order $n \geq 2$ is a finite probability space whose sample space is the set $\langle n \rangle := \{0,1,2, \cdots, n-1\}$ but that may have any probability distribution. Indexing by $\langle n \rangle$ rather than the standard $[n] := \{1, 2, \cdots, n\}$ simplifies many formulae in the sequel. We use the terms roll and side as synonyms for trial and outcome, respectively, motivated by the example of standard cubical dice. However, our dice often have sides with probability 0, so a better mental model is a spinner mounted over a circle divided into n arbitrary sectors. The language of dice is historical in our problem.

We index sides of dice by j and denote the probability of side j by $p_{\boldsymbol{d}}(j)$, omitting the $\boldsymbol{d}$ when possible. We will confound the die $\boldsymbol{d}$, the tautological random variable whose value on side j is j, and the die polynomial $\boldsymbol{d}(x) := \sum_{j=0}^{n-1} p(j)x^j$, which is the generating function of this random variable. For example, a standard fair cubical die has $\boldsymbol{d}(x) := \frac{1}{6}(1 + x + x^2 + x^2 + x^4 + x^5)$. As in this example, we will always write such polynomials with degrees increasing from left to right.

Definition 2.1. A die is semifair if

- **(a)** each $p(j)$ is either 0 or equal to $p(0)$, which must be nonzero,
- **(b)** it is palindromic: that is, $p(n - j - 1) = p(j)$.

Remark 2.2. A few remarks about semifairness are in order.

- **(a)** Henceforth, we abuse notation by rescaling semifair dice so that $p(0)$, the common value of the nonzero $p(j)$, is 1. Since the probability condition that the unscaled $p(j)$ sum to exactly

1 allows us to reserve the scaling, we lose nothing by assuming this. Doing so allows us to avoid denominators and be able to work with monic polynomials throughout.

(b) Se $\Psi_t(x) := 1 + x + \cdots + x^{t-1} = \left(\frac{1-x^t}{1-x}\right)$. The first form shows that $\Psi_t(x)$ is the polynomial of a fair die of order t –see the cubical example above—and the second that its roots are exactly the tth roots of unity, except for 1.

(c) A die $\boldsymbol{d}$ of order n is a semifair if and only if $\boldsymbol{d}(x)$ is obtained from $\Psi_n(x)$ by setting to 0 a palindromic set of the interior coefficients.

A sack $\boldsymbol{S}$ of size $m_{\boldsymbol{S}}$ is a set of independent dice $\boldsymbol{d}_i$ of orders $n_i \geq 2$ indexed by $i \in [m_{\boldsymbol{S}}]$. To simplify notation, we omit reference to $\boldsymbol{S}$ when it is understood and write, for example, m for $m_{\boldsymbol{S}}$. Such an $\boldsymbol{S}$ has a product sample space $\boldsymbol{J}$ indexed by rolls $\boldsymbol{j} = (j_1, j_2, \cdots, j_m) \in \prod_{i\in[m]}\langle n_i\rangle$ that carries, by independence, the product probability distribution $p(\boldsymbol{j}) = \prod_{i\in[\boldsymbol{m}]} p_{\boldsymbol{d}_i}(j_i)$.
On $\boldsymbol{J}$, we have independent random variables for each die $\boldsymbol{d}_i$ whose value on any roll $\boldsymbol{j}$ is j_i and whose generating function is thus the die polynomial $\boldsymbol{d}_i(x)$. We sum these to get the total random variable $\boldsymbol{T}(\boldsymbol{j}) := \sum_{i\in[m]} j_i$, which takes on the t values in $\langle t\rangle$, where $t - 1 := \sum_{i\in[m]}(n_i - 1)$.
Since the generating function of a sum of independent random variables is the product of the generating functions of its terms, the total $\boldsymbol{T}$ has generating function

$$\boldsymbol{T}(x) = \prod_{i\in[m]} \boldsymbol{d}_i(x) = \sum_{s=0}^{t-1}\left(\sum_{T(\boldsymbol{j})=s} p(\boldsymbol{j})\right)x^s.$$

For two standard dice, this is a shifted form of the familiar for totals:

$$1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + 5x^6 + 4x^7 + 3x^8 + 2x^9 + x^{10} = (1 + x + x^2 + x^3 + x^4 + x^5)^2.$$

A fair sack is simply one for which $\boldsymbol{T}(x) = \Psi_t(x)$.

Gasarch-Kruskal Theorem. A sack is fair if and only if

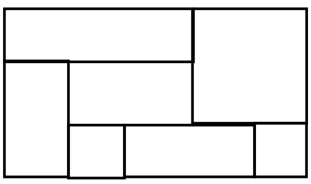
(a) each die in it is semifair,

(b) (Uniqueness of Totals) each total is obtained from a unique effective roll.

We first show that the dice $\boldsymbol{d}(x)$ in a fair sack must be semifair, which is the heart of the theorem. Palandromicity is easy. If the sack has order t, let ξ be a primitive tth root of unity. The irreducible real factors of $\Psi_t(x)$ are $x + 1$, when t is even, and $x^2 - (\xi^j + \xi^{-j})x + 1$, for $j \in \left[\left\lfloor\frac{t-1}{2}\right\rfloor\right]$. These are palindromic and $\boldsymbol{d}(t)$, being real, must be a product of them, so it is palindromic. The key step is the following.

Lemma 2.3. If $\boldsymbol{d}(x) := \boldsymbol{d}'(x) \cdot \boldsymbol{d}''(x)$ is semifair and both $\boldsymbol{d}'$ and $\boldsymbol{d}''$ are palindromic, then both $\boldsymbol{d}'$ and $\boldsymbol{d}''$ are semifair.
Given the lemma, an induction on the size s of a fair sack $\boldsymbol{S}$ shows semifairness of its dice. The case $s = 1$ is trivial. If $s \geq 2$, just take any two dice $\boldsymbol{d}'(x)$ and $\boldsymbol{d}''(x)$ and replace them with their product $\boldsymbol{d}(x)$, getting a fair sack of smaller size whose dice, in particular $\boldsymbol{d}(x)$, must inductively be semifair. Since we know already that $\boldsymbol{d}'$ and $\boldsymbol{d}''$ are palindromic, the lemma then shows that both are also semifair.

10) AMM 12005 [2018, 755] A tight m-by-n paving is a decomposition of an m-by-n rectangle into $m + n - 1$ rectangular tiles with integer sides such that each of the $m - 1$ horizontal lines and $n - 1$ vertical lines within the rectangle is part of the boundary of at least one tile. For example, one of the 1071 tight 3-by-5 pavings is pictured here:



Let $a_{m,n}$ denote the number of tight m-by-n pavings.

a) Determine $a_{3,n}$ as a function of n.
b) Show for $m \geq 3$ that $\lim_{n\to\infty} a_{m,n}/m^n$ exists, and compute its value.

11) Richard Stanley The number of 1's in the partitions of n is equal to the number of parts that appear at least once in a given partition of n, summed over all partitions of n.

12) The number of 1's in the partitions of n is equal to

$$\sum_{k=2}^{n+1} \phi(k) S_{n+1,k}^{(2)} .$$

(We denote by $S_{n,k}^{(r)}$ the number of k's in the partitions of n with the smallest part at least r.)

13) For $n \geq 0$, $p(n) = \frac{1}{2}\sum_{k=3}^{n+3} \phi(k)\, S_{n+3,k}^{(3)}$.

14) Tim Antonelli, AMM May 2019 Pick an integer $n > 1$. Write down every multiset λ of the positive integers that add to n but do not include 1. Apply the function $f(\lambda)$ by taking the products of the parts, multiplying by the factorials of the parts' multiplicities k_i, and talking the reciprocal, as shown in Table 1 for $n = 9$. Finally, add together these $f(\lambda)$. Do you recognize the result? What about is reciprocal? The sum $\sum_\lambda f(\lambda)$ turns out to be the nth Taylor series approximation for $e^{-1} = 0.367879\ldots$.
Table 1. Arithmetic on a restricted partition of $n = 9$.

λ	Product	k_i	$f(\lambda)$
(9)	9	1	$\frac{1}{9}$
$(7,2)$	$7 \cdot 2$	$1,1$	$\frac{1}{7 \cdot 2}$
$(6,3)$	$6 \cdot 3$	$1,1$	$\frac{1}{6 \cdot 3}$
$(5,4)$	$5 \cdot 4$	$1,1$	$\frac{1}{5 \cdot 4}$
$(5,2,2)$	$5 \cdot 2^2$	$1,2$	$\frac{1}{5 \cdot 2^2 \cdot 2!}$
$(4,3,2)$	$4 \cdot 3 \cdot 2$	$1,1,1$	$\frac{1}{4 \cdot 3 \cdot 2}$
$(3,3,3)$	3^3	3	$\frac{1}{3^3 \cdot 3!}$
$(3,2,2,2)$	$3 \cdot 2^3$	$1,3$	$\frac{1}{3 \cdot 2^3 \cdot 3!}$
			$\sum_\lambda f(\lambda) \approx 0.367879$

Matrizes

Equações
15) AMM 12028 [2018, 276] We have n coins, where $n = d + p + q$ for positive integers d, p and q. Suppose that whenever any d of the coins are removed, the rest can be split into two sets of size p and q that balance when placed on a balance with arm lenghts q and p, respectively. That is, q times the weight of the p coins equals p times the weight of the q coins. Must all n coins have the same weight?

Determinantes/Contagem
16) Young Tableaux
In order to count the Young tableaux of a given shape, we need to introduce the function $\Delta(x_1, \cdots, x_m)$ defined by

$$\Delta(x_1, \cdots, x_m) := \prod_{1 \le i < j \le m} (x_i - x_j).$$

(Note that Δ is the value of the Vandermode determinant.)
Let

$$g(x_1, \cdots, x_m; y) := x_1 \Delta(x_1 + y, x_2, \cdots, x_m) \\ + x_2 \Delta(x_1, x_2 + y, \cdots, x_m) + \cdots + x_m \Delta(x_1, x_2, \cdots, x_m + y).$$

Then

$$g(x_1, \cdots, x_m; y) = \left(x_1 + \cdots + x_m + \binom{m}{2} y\right) \Delta(x_1, \cdots, x_m).$$

We introduce a function f defined on all m-tuples $(n_1, \cdots, n_m)$, $m \ge 1$, with the following properties:

$$f(n_1, \cdots, n_m) = 0 \text{ unless } n_1 \ge n_2 \ge \cdots \ge 0;$$
$$f(n_1, \cdots, n_m, 0) = f(n_1, \cdots, n_m);$$
$$f(n_1, \cdots, n_m) = f(n_1 - 1, n_2, \cdots, n_m) + \\ f(n_1, n_2 - 1, \cdots, n_m) + \cdots + f(n_1, n_2, \cdots, n_m - 1),$$
$$\text{if } n_1 \ge n_2 \ge \cdots \ge n_m \ge 0;$$
$$f(n) = 1 \quad \text{if } n \ge 0.$$

The number of Young tableaux that have shape $(n_1, \cdots, n_m)$ satisfies

$$f(n_1, \cdots, n_m) = \frac{\Delta(n_1 + m - 1, n_2 + m - 2, \cdots, n_m) n!}{(n_1 + m - 1)!\,(n_2 + m - 2)! \cdots n_m!},$$

and in fact this formula for f is correct if $n_1 + m - 1 \ge n_2 + m - 2 \ge \cdots \ge n_m$.

Determinantes/Contagem
17) Catalan
The Catalan numbers C_n are given by the formula

$$C_n = \frac{1}{n+1} \binom{2n}{n}, \qquad n \ge 0.$$

The Catalan sequence begins

$$1, 1, 2, 5, 14, 42, 132, 429, 1430, 5862, \cdots.$$

Catalan numbers occur in many combinatorial situations. For instance, C_n is the number of paths in the x-y coordinate system that start at $(0,0)$ and stop at (n,n), moving at each step either one unit right or one unit up and never crossing the line $y = x$.
For any $n \ge 1$, form the $n \times n$ matrix whose (i,j)th entry is the Catalan number C_{i+j-2}. Prove that for every n the determinant of the matrix is 1.