IMC 2020 Online

Day 1, July 26, 2020

Problem 1. Let n be a positive integer. Compute the number of words w (finite sequences of letters) that satisfy all the following three properties:

(1) w consists of n letters, all of them are from the alphabet $\{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}\}$;

(2) w contains an even number of letters $\mathtt{a}$;

(3) w contains an even number of letters $\mathtt{b}$.

(For example, for $n = 2$ there are 6 such words: $\mathtt{aa}$, $\mathtt{bb}$, $\mathtt{cc}$, $\mathtt{dd}$, $\mathtt{cd}$ and $\mathtt{dc}$.)

Armend Sh. Shabani, University of Prishtina

Problem 2. Let A and B be $n \times n$ real matrices such that

$$\mathrm{rk}(AB - BA + I) = 1$$

where I is the $n \times n$ identity matrix.

Prove that

$$\mathrm{trace(ABAB)} - \mathrm{trace(A^2B^2)} = \frac{1}{2}\mathrm{n(n-1)}.$$

($\mathrm{rk}(M)$ denotes the rank of matrix M, i.e., the maximum number of linearly independent columns in M. trace(M) denotes the trace of M, that is the sum of diagonal elements in M.)

Rustam Turdibaev, V. I. Romanovskiy Institute of Mathematics

Problem 3. Let $d \geq 2$ be an integer. Prove that there exists a constant $C(d)$ such that the following holds: For any convex polytope $K \subset \mathbb{R}^d$, which is symmetric about the origin, and any $\varepsilon \in (0, 1)$, there exists a convex polytope $L \subset \mathbb{R}^d$ with at most $C(d)\varepsilon^{1-d}$ vertices such that

$$(1 - \varepsilon)K \subseteq L \subseteq K.$$

(For a real α, a set $T \subset \mathbb{R}^d$ with nonempty interior is a *convex polytope with at most α vertices*, if T is a convex hull of a set $X \subset \mathbb{R}^d$ of at most α points, i.e., $T = \{\sum_{x \in X} t_x x \mid t_x \geq 0, \sum_{x \in X} t_x = 1\}$. For a real λ, put $\lambda K = \{\lambda x \mid x \in K\}$. A set $T \subset \mathbb{R}^d$ is *symmetric about the origin* if $(-1)T = T$.)

Fedor Petrov, St. Petersburg State University

Problem 4. A polynomial p with real coefficients satisfies the equation $p(x + 1) - p(x) = x^{100}$ for all $x \in \mathbb{R}$. Prove that $p(1 - t) \geqslant p(t)$ for $0 \leqslant t \leqslant 1/2$.

Daniil Klyuev, St. Petersburg State University

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Day 2, July 27, 2020

Problem 5. Find all twice continuously differentiable functions $f : \mathbb{R} \to (0, +\infty)$ satisfying

$$f''(x)f(x) \geq 2(f'(x))^2$$

for all $x \in \mathbb{R}$.

Karen Keryan, Yerevan State University & American University of Armenia, Yerevan

Problem 6. Find all prime numbers p for which there exists a unique $a \in \{1, 2, \ldots, p\}$ such that $a^3 - 3a + 1$ is divisible by p.

Géza Kós, Loránd Eötvös University, Budapest

Problem 7. Let G be a group and $n \geq 2$ be an integer. Let H_1 and H_2 be two subgroups of G that satisfy

$$[G : H_1] = [G : H_2] = n \quad \text{and} \quad [G : (H_1 \cap H_2)] = n(n-1).$$

Prove that H_1 and H_2 are conjugate in G.

(Here $[G : H]$ denotes the *index* of the subgroup H, i.e. the number of distinct left cosets xH of H in G. The subgroups H_1 and H_2 are *conjugate* if there exists an element $g \in G$ such that $g^{-1}H_1g = H_2$.)

Ilya Bogdanov and Alexander Matushkin, Moscow Institute of Physics and Technology

Problem 8. Compute

$$\lim_{n\to\infty} \frac{1}{\log\log n} \sum_{k=1}^{n} (-1)^k \binom{n}{k} \log k.$$

(Here log denotes the natural logarithm.)

Fedor Petrov, St. Petersburg State University