Grandes Ideias e Problemas Muito Legais de Combinatória e Probabilidades

Semana Olímpica 2021 – Teresina PI – Prof. Edmilson Motta

Os problemas de 1 a 8 estão no capítulo "Probability In Your Head" escrito por Peter Winkler para o livro "The Mathematics of Various Entertaining Subjects – Volume 3 – The Magic of Mathematics", editado por Jennifer Beineke e Jason Rosenhouse.
Os problemas de 9 a 11 estão no capítulo "Should You Be Happy" escrito por Peter Winkler para o livro "The Mathematics of Various Entertaining Subjects – Research in Recreational Mathematics", editado por Jennifer Beineke e Jason Rosenhouse.
Os problemas de 12 a 29 são do livro Conversational Problem Solving escrito por Richard P. Stanley.

1) *Flying Saucers*
A fleet of saucers from planet Xylofon has been sent to bring back the inhabitants of a certain apartment building, for exhibition in the planet zoo. The earthlings therein constitute 11 men and 14 women.
Saucers arrive one at a time and randomly beam people up. However, owing to the Xylofonians' strict sex separation policy, a saucer cannot take off with humans of both sexes. Consequently, a saucer will continue beaming people up until it acquires a member of a second sex; that human is immediately beamed back down, and the saucer takes off with whomever it already has on board. Another saucer then swoops in, again beaming up people at random until it gets one of a new gender, and so forth, until the building is empty. What is the probability that the last person beamed up is a woman?

2) *Points on a Circle*
Three points are chosen at random on a circle. What is the probability that there is a semicircle of that circle containing all three?

3) *Meet the Williams Sisters*
Some tennis fans get excited when Venus and Serena Williams meet in a tournament. The likelihood of that happening normally depends on seeding and talent, so let us instead assume an idealized elimination tournament of 64 players, each as likely to win as to lose any given match, with bracketing chosen uniformly at random. What is the probability that the Williams sisters get to play each other?

4) *Service options*
You are challenged to a short tennis match, with the winner to be the first player to win four games. You get to serve first. But there are options for determining the sequence in which the two of you serve:
1. Standard: Serve alternates (you, her, you, her, you, her, you).
2. Volleyball style: The winner of the previous game serves the next one.
3. Reverse volleyball style: The winner of the previous game receives in the next one.
Which option should you choose? You may assume it is to your advantage to serve. You may also assume that the outcome of any game is independent of when the game is played and of the outcome of any previous game.

5) *Who Won the Series?*
Two evenly-matched teams meet to play a best-of-seven World Series of baseball games. Each team has the same small advantage when playing home. As usual, one team (say, Team A) plays games 1 and 2 at home, and, if necessary, plays games 6 and 7 at home. Team B plays 3,4 and, if needed, 5 at home.
You go to a conference in Europe and return to find that the series is over, and six games were played. Which team is more likely to have won the series?

6) *Random Rice*
You go to the grocery store needing 1 cup of rice. When you push the button on the machine, it dispenses a uniformly random amount of rice between nothing and 1 cup. On average, how many times do you have to push the button to get (at least) a cupful?

7) *Six with No Odds*
On average, how many times do you need to roll a dice to get a 6, given that you do not roll any odd number *en route*? (Hint: The answer is not 3).

8) *Getting the Benz*
Your rich aunt has died and left her beloved 1955 Mercedes-Benz 300 SL Gullwing to either you or one of your four siblings, according to the following stipulations. Each of the five of you will privately write "1", "2", or "3" on a slip of paper. The slips are put into a bowl to be examined by the estate lawyers, who will award the car to heir whose number was not entered by anyone else. (If there is no such heir, or more than one, the procedure is repeated.)
For example, if the bowl contains one 1, two 2s and two 3s, the one who put in the 1 gets the car.
After stewing and then shrugging your shoulders, you write a 1 on your slip and put it in the bowl. What the heck, a $1/5$ chance at this magnificent vehicle is not to be sneezed at! But just before the bowl is passed to the lawyers, you get a sneak peek and can just make out, among the five slips of paper, one 1 (which may or may not be yours), one 2, and one 3.
Should you be happy, unhappy, or indifferent to this information?

9) *Back to Baseball*
You are a rabid baseball fan and, miraculously, your team has won the pennant – thus, it gets to play in the World Series. Unfortunately, the opposition is a superior team whose probability of winning any given game against your team is 60%.
Sure enough, your team loses the first game in the best-of-seven series, and you are so unhappy that you drink yourself into a stupor. When you regain consciousness, you discover that two more games have been played.
You run out into the street and gran the first passer-by. "What happened in games two and three of the World Series?"
"They were split", she says. "One game each."
Should you be happy?

10) *A Chess Problem, of Sorts*
You want to join a certain chess club, but admission requires that you play three games against Ioana, the current club champion, and win two in a row.
Since it is an advantage to play the white pieces (which move first), you alternate playing white and black.

A coin is flipped, and the result is that you will be white in the first and third games, black in second. Should you be happy?

11) *Coin-Flipping and Dishwashing*
You and your spouse flip a coin to see who washes the dishes each evening. "Heads" he washes, "tails" you wash.
Tonight, he tells you he is imposing a different scheme. You flip the coin thirteen times, then he flips it twelve times. If you get more heads than he does, he washes; if you get the same number of heads of fewer, you wash.
Should you be happy?

12) Given positive integers n and k, how many k-tuples $(S_1, \dots, S_k)$ of subsets of $\{1, 2, \dots, n\}$ are there such that $S_1 \cap S_2 \cap \dots \cap S_k = \emptyset$?

13) Let w be a random permutation (uniform distribution) of $1, 2, \dots, n$ and fix $1 \leq k \leq n$. What is the probability that in the disjoint cycle decomposition of w, the length of the cycle containing 1 is k? In other words, what is the probability that k is the least positive integer for which $w^k(1) = 1$?

14) Given a random permutation w, what is the probability that 1 and 2 are in the same cycle?

15) Choose n real numbers uniformly and independently from the interval $[0,1]$. What is the expected value of $\min_i x_i$, the minimum value of $x_1, \dots, x_n$?

16) Given integers $m, n \geq 0$, evaluate the integral $\int_0^1 x^m (1-x)^n \, dx$.

17) (Putnam1984) $\iiint_R x^1 y^9 z^8 w^4 \, dx \, dy \, dz$, where R is the region of $\mathbb{R}^3$ defined by $x, y, z \geq 0$ and $x + y + z \leq 1$, and where $w = 1 - x - y - z$.

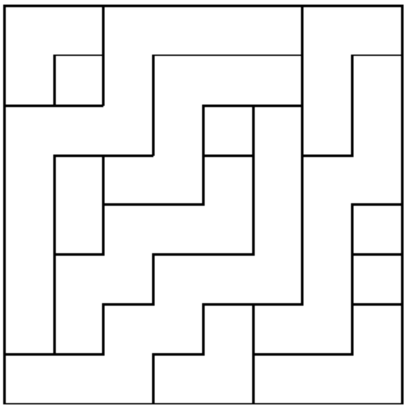
18) Choose n points at random (uniformly and independently) on the circumference of a circle. Find the probability p_n that all the points lie on a semicircle. For instance, $p_1 = p_2 = 1$.

19) Choose four points at random (uniformly and independently) on the surface of a sphere. What is the probability that the center of the sphere is contained in the convex hull of the four points?

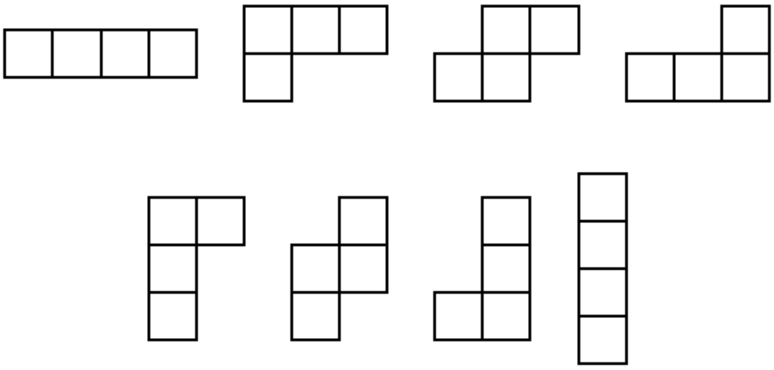
20) Passengers $P_1, \dots, P_n$ enter a plane with n seats. Each passenger has a different assigned seat. The first passenger sits in the wrong seat. Thereafter, each passenger either sits in their seat if unoccupied or otherwise sits in a random unoccupied seat. What is the probability that the last passenger P_n sits in his own seat?

21) Let $x_1, x_2, \dots, x_n$ be n points (in that order) on the circumference of a circle. Rebecca starts at the point x_1 and walks to one of the two neighboring points with probability $1/2$ for each. She continues to walk in this way, always moving from the present point to one of the two neighboring points with probability $1/2$ for each. Find the probability p_i that the point x_i is the last unvisited point. In other words, find the probability p_i that when x_i is visited for the first time, all other points will have already been visited. For instance, $p_i = 0$ (when $n > 1$), since x_1 is the first of the n points to be visited.

22) A snake on an $m \times n$ chessboard is a nonempty subset S of the squares of the board obtained as follows. Start at one of the squares and continue walking one step up or to the right, stopping at any time. The squares visited are the squares of the snake. Here is an example of the 8 x 8 chessboard covered with disjoint snakes:



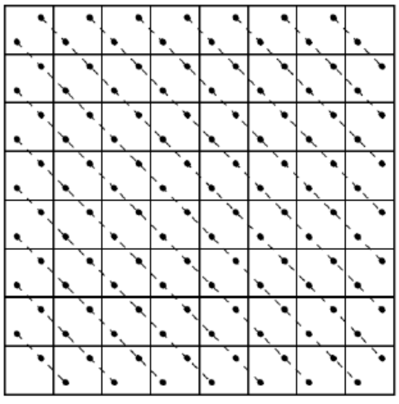
If we consider two snakes equivalent if they are translations of each other, then here are the eight inequivalent snakes of size four (that is, with four squares):



How many inequivalent snakes have k squares?

23) In how many ways can we tile an $m \times n$ rectangle with snakes? There is no restriction on the size or number of snakes.

Dica:



24) Show that the number of ways to tile an $n \times n$ board with n snakes of size n is $n!$.

25) Recall that an increasing subsequence of a permutation $a_1 a_2 \dots a_n$ of $[n]$ (or more generally, of any sequence of integers) is a sequence $a_{i_1} < a_{i_2} < \cdots < a_{i_k}$ such that $1 \leq i_1 < i_2 < \cdots < i_k \leq n$. What is the expected number of increasing subsequences of length k (where $1 \leq k \leq n$) of a random permutation (uniform distribution) w?

26) An evil warden is in charge of 100 prisoners (all with different names). He puts a row of 100 boxes in a room. Inside each box is a name of a different prisoner. The prisoners enter the room one at a time. Each prisoner must open 50 of the boxes, one at a time. If any of the prisoners does not see his or her own name, then they are all killed. The prisoners may have a discussion before the first prisoner enters the room with the boxes, but after that there is no further communication. A prisoner may not leave a message of any kind for another person. In particular, all the boxes are shut once a prisoner leaves the room. If all the prisoners choose 50 boxes at a random, then each has a success probability of $1/2$, so the probability that they are not killed is 2^{-100}, not such good odds. Is there a strategy that will increase the chances of success? What is the best strategy?

27) If a permutation w is chosen at random from $\mathfrak{S}_n$ with the uniform distribution, what is the expected number $f(n)$ of cycles of w in its disjoint cycle decomposition? For instance, $\mathfrak{S}_3$ has one permutation with three cycles, three with two cycles, and two with one cycle, so $f(3) = \frac{1}{6}(1.3 + 3.2 + 2.1) = \frac{11}{6}$.

28) Why is the expected number of k cycles equal to $1/k$? We want to avoid induction and generating functions.

29) What is the expected length $L(n)$ of the longest cycle in a random permutation (uniformly distributed as usual) $w \in \mathfrak{S}_n$?