IMC 2021 Online

First Day, August 3, 2021

Problem 1. Let A be a real $n \times n$ matrix such that $A^3 = 0$.

(a) Prove that there is a unique real $n \times n$ matrix X that satisfies the equation

$$X + AX + XA^2 = A.$$

(b) Express X in terms of A.

(10 points)

Problem 2. Let n and k be fixed positive integers, and let a be an arbitrary non-negative integer. Choose a random k-element subset X of $\{1, 2, \dots, k+a\}$ uniformly (i.e., all k-element subsets are chosen with the same probability) and, independently of X, choose a random n-element subset Y of $\{1, \dots, k+n+a\}$ uniformly.

Prove that the probability

$$\mathsf{P}\Big(\min(Y) > \max(X)\Big)$$

does not depend on a.

(10 points)

Problem 3. We say that a positive real number d is *good* if there exists an infinite sequence $a_1, a_2, a_3, \ldots \in (0, d)$ such that for each n, the points $a_1, \dots, a_n$ partition the interval $[0, d]$ into segments of length at most $1/n$ each. Find

$$\sup\Big\{d \;\Big|\; d \text{ is good}\Big\}.$$

(10 points)

Problem 4. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Suppose that for every $\varepsilon > 0$, there exists a function $g : \mathbb{R} \to (0, \infty)$ such that for every pair (x, y) of real numbers,

$$\text{if} \quad |x - y| < \min\Big\{g(x), g(y)\Big\}, \quad \text{then} \quad \Big|f(x) - f(y)\Big| < \varepsilon.$$

Prove that f is the pointwise limit of a sequence of continuous $\mathbb{R} \to \mathbb{R}$ functions, i.e., there is a sequence $h_1, h_2, \ldots$ of continuous $\mathbb{R} \to \mathbb{R}$ functions such that $\lim\limits_{n\to\infty} h_n(x) = f(x)$ for every $x \in \mathbb{R}$.

(10 points)

IMC 2021 Online

Second Day, August 4, 2021

Problem 5. Let A be a real $n \times n$ matrix and suppose that for every positive integer m there exists a real symmetric matrix B such that

$$2021B = A^m + B^2.$$

Prove that $|\det A| \leq 1$.

(10 points)

Problem 6. For a prime number p, let $\mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z})$ be the group of invertible 2×2 matrices of residues modulo p, and let S_p be the symmetric group (the group of all permutations) on p elements. Show that there is no injective group homomorphism $\varphi : \mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z}) \to S_p$.

(10 points)

Problem 7. Let $D \subseteq \mathbb{C}$ be an open set containing the closed unit disk $\{z \ : \ |z| \leq 1\}$. Let $f : D \to \mathbb{C}$ be a holomorphic function, and let $p(z)$ be a monic polynomial. Prove that

$$\left|f(0)\right| \leq \max_{|z|=1} \left|f(z)p(z)\right|.$$

(10 points)

Problem 8. Let n be a positive integer. At most how many distinct unit vectors can be selected in $\mathbb{R}^n$ such that from any three of them, at least two are orthogonal?

(10 points)