Homotetia

Prof.Luciano Monteiro de Castro
lucianogmcastro@gmail.com
@matematicamentexyz

Problemas

1. Two lines l_1 and l_2 and a point A are given. Pass a line l through A so that the segment BC cut off by l_1 and l_2 satisfies $AB : AC = m : n$.

2. (a) A circle S and a point A on it are given. Find the locus of the midpoints of all the chords through A.

(b) A circle S and three points A, B, and C on it are given. Draw a chord AX that bisects the chord BC.

3. Let two tangent circles R and S be given. Let l be a line through the point M of tangency, and let this line meet R in a second point A and meet S in a second point B. Show that the tangent to R at A is parallel to the tangent to S at B.

4. Let R and S be two disjoint circles, neither inside the other. Let m be a common tangent to R and S and assume that R and S are both on the same side of m. Let n be another common tangent, with R and S both on the same side of n. Let M be the point of intersection of m and n. Let l be a line through M meeting R in points A and B, and meeting S in points C and D. Finally, let E be the point of tangency of m and R, and let F be the point of tangency of m and S. Prove that:

(a) Triangle ABE is similar to triangle CDF.

(b) The ratio of the areas of triangles ABE and CDF is equal to the square of the ratio of the radii of R and S.

(c) The line determined by the points of intersection of the medians of triangles ABE and CDF passes through the point M.

5. Let $ABCD$ be a trapezoid whose sides AD and BC, extended, meet in a point M; let N be the point of intersection of the diagonals AC and BD. Prove that:

(a) The circles R and S circumscribed about triangles ABM and DCM are tangent.

(b) The circles R_1 and S_1 circumscribed about triangles ABN and CDN are tangent.

(c) The ratio of the radii of R_1 and S_1 is equal to the ratio of the radii of R and S.

6. (a) Using the two parallel sides AB and CD of trapezoid $ABCD$ as bases construct equilateral triangles ABE and CDF. These triangles should each be on the same side of the base (that is, if we regard AB and CD as being horizontal, then either both triangles are constructed above the base, or both are constructed below the base). Prove that the line EF passes through the point of intersection of the extensions of the two nonparallel sides of the trapezoid.

(b) On the parallel sides AB and CD of the trapezoid, construct squares exterior to the trapezoid. Prove that the line joining their center passes through the point of intersection of the diagonals of the trapezoid.

7. Prove that the line joining the midpoints of the two parallel sides of a trapezoid passes through the point of intersection of the extensions of the other two sides, as well as through the point of intersection of the diagonals.

8. (a) Two concentric circles S_1 and S_2 are given. Draw a line l meeting these circles consecutively in points A, B, C, D such that $AB = BC = CD$. (b) Three concentric circles S_1, S_2 and S_3 are given. Draw a line l meeting S_1, S_2 and S_3 in order in points A, B, and C such that $AB = BC$. (c) Four concentric circles S_1, S_2, S_3 and S_4 are given. Draw a line l meeting S_1, S_2, S_3 and S_4 respectively in points A, B, C, D such that $AB = CD$.

9. (a) Inscribe a square in a given triangle ABC so that two vertices lie on the base AB, and the other two lie on the sides AC and BC.

(b) In a given triangle ABC inscribe a triangle whose sides are parallel to three given lines l_1, l_2 and l_3.

10. (a) Two lines l_1 and l_2 are given together with a point A on l_1 and a point B on l_2. Draw segments AM_1 and BM_2 on the lines l_1 and l_2 having a given ratio $AM_1/BM_2 = m$, and through the points M_1 and M_2 pass lines parallel to two given lines l_3 and l_4 (Figure 11). Find the locus of the points of intersection of these lines.

(b) A polygon $A_1A_2 \cdots A_n$ varies in such a way that its sides remain parallel to given directions, and the vertices $A_1, A_2, \cdots, A_{n-1}$ move on given lines $l_1, l_2, \cdots, l_{n-1}$. Find the locus of the vertex A_n.

(c) In a given polygon inscribe another polygon whose sides are parallel to given lines.

11. Let a "hinged" parallelogram $ABCD$ be given. More precisely, the lengths of the sides are fixed, and vertices A and B are fixed, but vertices C and D are movable (Figure 12). Prove that, as C and D move, the point Q of intersection of the diagonals moves along a circle.

12. (a) Let the inscribed circle S of $\triangle ABC$ meet BC at the point D. Let the escribed circle which touches BC and the extensions of sides AB and AC meet BC at the point E. Prove that AE meets S at the point D_1 diametrically opposite to D.

(b) Construct triangle ABC, given the radius r of the inscribed circle, the altitude $h = AP$ on side BC, and the difference $b - c$ of the two other sides.

13. Construct a circle S

(a) tangent to two given lines l_1 and l_2 and passing through a given point A.

(b) passing through two given points A and B and tangent to a given line l.

(c) tangent to two given lines l_1 and l_2 and to a given circle S.

14. (a) Prove that the point M of intersection of the medians of triangle ABC, the center O of the circumscribed circle, and the point H of intersection of the altitudes lie on a line, and that $HM/MO = 2/1$.

(b) Prove that the three lines through the midpoints of the sides of a triangle and parallel to the bisectors of the opposite angles meet in a single point.

(c) Prove that the lines joining the vertices of triangle ABC to the points where the opposite sides are tangent to the corresponding escribed circles meet in a single point J. This point is collinear with the point M of intersection of the medians and the center Z of the inscribed circle, and $JM/MZ = 2/1$.

15. Inscribe a triangle ABC in a given circle S, if the vertex A and the point H of intersection of the altitudes are given.

16. Given a circle S. What is the locus of the points of intersection of the (a) medians, (b) altitudes, of all possible acute angled triangles inscribed in S; of all right angled triangles inscribed in S; of all obtuse angled triangles inscribed in S?

17. In the triangle $ABC, AB = AC$. A circle is tangent internally to the circumcircle of the triangle and also to AB, AC at P, Q respectively. Prove that the midpoint of PQ is the center of the incircle of the triangle.

18. Three circles of equal radius have a common point O and lie inside a given triangle. Each circle touches a pair of sides of the triangle. Prove that the incenter and the circumcenter of the triangle are collinear with the point O.

19. A non-isosceles triangle $A_1A_2A_3$ has sides a_1, a_2, a_3 with a_i opposite A_i. M_i is the midpoint of side a_i and T_i is the point where the incircle touches side a_i. Denote by S_i the reflection of T_i in the interior bisector of angle A_i. Prove that the lines M_1S_1, M_2S_2 and M_3S_3 are concurrent.

20. Theorem on the three centers of similarity. Let the figure F_1 be centrally similar to the figure F with similarity center O_1, and let it also be centrally similar to the figure F' with similarity center O_2. If O_1 does not coincide with O_2, then the line O_1O_2 passes through the center of similarity O of the figures F and F', or is parallel to the direction of the translation carrying F into F'. If O_1 coincides with O_2, then O_1 is the center of similarity also for F and F'.

If O_1 is different from O_2 then the line O_1O_2 is called the axis of similarity of the three figures F, F_1 and F'; if O_1 coincides with O_2 then this point is called the center of similarity of the figures F, F_1 any F'.

Usually the theorem on the three centers of similarity is formulated somewhat less precisely as follows: *the three centers of similarity of three pairwise centrally similar figures lie on a line.*

Quase todos os problemas desta lista foram retirados do excelente livro "Geometric Transformations II", de I. M Yaglom (MAA, New Mathematical Library ,21), publicado em 1968. Disponível em www.ams.org.