OW 2023 – N2

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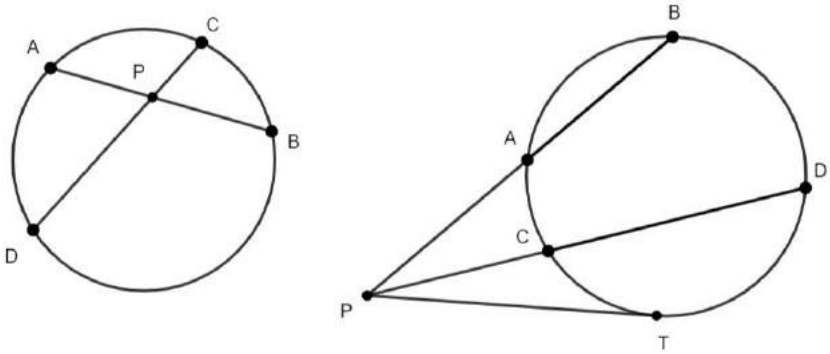
Part 1

Theorem 1 (Chord Theorem – "Power of a Point"). Let Γ be a circle, and P a point. Let a line through P meet Γ at points A and B, and let another line through P meet Γ at points C and D. Then

$$PA \cdot PB = PC \cdot PD$$

If P lies outside Γ and we draw PT tangent to Γ at T, then

$$PA \cdot PB = PC \cdot PD = PT^2$$



Proof.
For the first case, we have $\angle PDA = \angle PBC$ (arc AC) and $\angle DPA = \angle BPC$ (opposite at P). So $\Delta PDA \sim \Delta PBC(AA)$ and

$$\frac{PA}{PC} = \frac{PD}{PB} \Leftrightarrow PA \cdot PB = PC \cdot PD$$

In the second case, we also have $\Delta PDA \sim \Delta PBC(AA)$ and $PA \cdot PB = PC \cdot PD$.

For the other part we have $\angle PDT = \angle PTC$ (arc CT), $\angle DPT = \angle TPC$ and $\Delta PDT \sim \Delta PTC$ (AA). The ratio of the sides gives us

$$\frac{PT}{PC} = \frac{PD}{PT} \Leftrightarrow PT^2 = PC \cdot PD$$

Theorem 2 (Converse to power of a point). Let A, B, C, D be four distinct points. Let lines AB and CD intersect at P. Assume that either
(1) P lies on both line segments AB and CD, or
(2) P lies on neither line segments.
Then A, B, C, D are concyclic if and only if $PA \cdot PB = PC \cdot PD$

Proof.
Suppose that P lies on both line segments AB and CD. We have $\angle DPA = \angle BPC$ (opposite at P) and

$$PA \cdot PB = PC \cdot PD \Leftrightarrow \frac{PA}{PC} = \frac{PD}{PB} \Leftrightarrow \Delta PDA \sim \Delta PBC \Leftrightarrow \angle PDA = \angle PBC$$

This occur iff A, B, C, D are concyclic.
Case (2) is analogous.

Problems

1. (AMC/2020-12B) In unit square $ABCD$ the inscribed circle ω intersects CD at M and AM intersects ω at a point P different from M. What is AP?

(A) $\frac{\sqrt{5}}{12}$ (B) $\frac{\sqrt{5}}{10}$ (C) $\frac{\sqrt{5}}{9}$ (D) $\frac{\sqrt{5}}{8}$ (E) $\frac{2\sqrt{5}}{15}$

2. (AIME I/2019) In convex quadrilateral $KLMN$ side MN is perpendicular to diagonal KM, side KL is perpendicular to diagonal LN, $MN = 65$, and $KL = 28$. The line through L perpendicular to side KN intersects diagonal KM at O with $KO = 8$. Find MO.

3. (Brazil/2013) Let Γ be a circle and A a point outside Γ. The tangent lines to Γ through A touch Γ at B and C. Let M be the midpoint of AB. The segment MC meets Γ again at D and the line AD meets Γ again at E. Given that $AB = a$, $BC = b$, compute CE in terms of a and b.

4. (USAMO/1998) Let C_1 and C_2 be concentric circles, with C_2 in the interior of C_1 . Let A be a point on C_1 and B a point on C_2 such that AB is tangent to C_2. Let C be the second point of intersection of AB and C_1 , and let D be the midpoint of AB. A line passing through A intersects C_2 at E and F in such a way that the perpendicular bisectors of DE and CF intersect at a point M on AB. Find, with proof, the ratio AM/MC.

5. (Russia/2012) Consider the parallelogram $ABCD$ with obtuse angle A. Let H be the foot of perpendicular from A to the side BC. The median from C in triangle ABC meets the circumcircle of triangle ABC at the point K. Prove that points K, H, C and D lie on the same circle.

6. (IMO 2000) Two circles Γ_1 and Γ_2 intersect at M and N. Let ℓ be the common tangent to Γ_1 and Γ_2 so that M is closer to ℓ than N is. Let ℓ touch Γ_1 at A and Γ_2 at B. Let the line through M parallel to ℓ meet the circle Γ_1 again at C and the circle Γ_2 again at D. Lines CA and DB meet at E; lines AN and CD meet at P; lines BN and CD meet at Q. Show that $EP = EQ$.

Part 2

Definition (Power of a Point) The power of a point P with respect to a circle Γ of center O and radius r is defined by

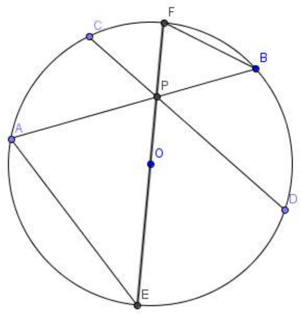
$$Pot_{\Gamma}P = PO^2 - r^2$$

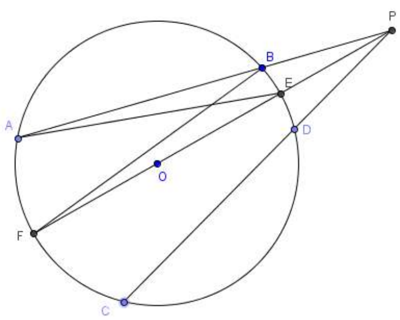
Theorem 3. If P is inside the circle Γ and a line through P cuts Γ at A and B, then

$$PA \cdot PB = -Pot_{\Gamma}P$$

If P is outside the circle Γ, a line through P cuts Γ at A and B and a line through P is tangent to Γ at T, then

$$PA \cdot PB = PT^2 = Pot_{\Gamma}P$$





Proof.
The line OP cross the circle on points E and F.
On the first case we have $PE = r + PO$ and $PF = r - PO$. By the Chrod Theorem

$$PA \cdot PB = PE \cdot PF = (r + PO)(r - PO) = r^2 - PO^2 = -Pot_{\Gamma}P$$

The second case is analogous.

Problems

7. (AMC/2013-10A) In $\Delta ABC, AB = 86$, and $AC = 97$. A circle with center A and radius AB intersects BC at points B and X. Moreover BX and CX have integer lengths. What is BC?

(A) 11 (B) 28 (C) 33 (D) 61 (E) 72

8. (AIME I/2019) Let AB be a chord of a circle ω, and let P be a point on the chord AB. Circle ω_1 passes through A and P and is internally tangent to ω. Circle ω_2 passes through B and P and is internally tangent to ω. Circles ω_1 and ω_2 intersect at points P and Q. Line PQ intersects ω at X and Y. Assume that $AP = 5$, $PB = 3$, $XY = 11$, and $PQ^2 = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

9. (Euler's relation) In a triangle with circumcenter O, incenter I, circumradius R, and inradius r, prove that

$$OI^2 = R^2 - 2Rr$$

10. (Tuymaada/2012) Point P is taken in the interior of the triangle ABC, so that

$$\angle PAB = \angle PCB = \frac{1}{4}(\angle A + \angle C)$$

Let L be the foot of the angle bisector of $\angle B$. The line PL meets the circumcircle of ΔAPC at point Q. Prove that QB is the angle bisector of $\angle AQC$.

11. (China/2013) Two circles K_1 and K_2 of different radii intersect at two points A and B, let C and D be two points on K_1 and K_2, respectively, such that A is the midpoint of the segment CD. The extension of DB meets K_1 at another point E, the extension of CB meets K_2 at another point F. Let l_1 and l_2 be the perpendicular bisectors of CD and EF, respectively.

i) Show that l_1 and l_2 have a unique common point (denoted by P).

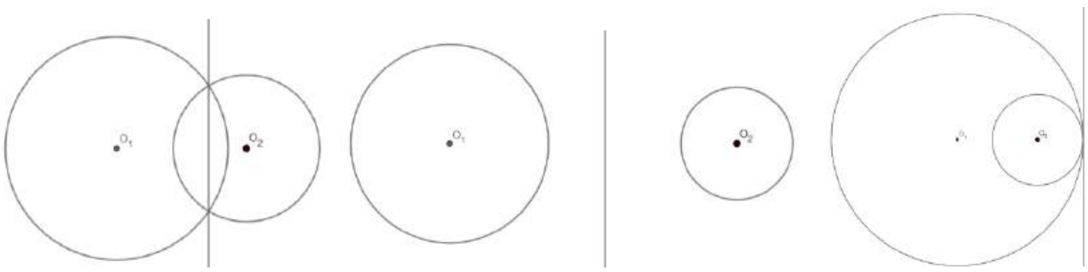
ii) Prove that the lengths of CA, AP and PE are the side lengths of a right triangle.

12. (IMO Shortlist/2011) Let $A_1A_2A_3A_4$ be a non-cyclic quadrilateral. Let O_1 and r_1 be the circumcenter and the circumradius of the triangle $A_2A_3A_4$. Define O_2, O_3, O_4 and r_2, r_3, r_4 in a similar way. Prove that

$$\frac{1}{O_1A_1^2 - r_1^2} + \frac{1}{O_2A_2^2 - r_2^2} + \frac{1}{O_3A_3^2 - r_3^2} + \frac{1}{O_4A_4^2 - r_4^2} = 0$$

Part 3

Theorem 4. (Radical Axis) Given two circles Γ_1 and Γ_2 with different centers, the locus of the points P on the plane such that the power of P with respect to Γ_1 is equal the power of P with respect to Γ_2 ($Pot_{\Gamma_1}P = Pot_{\Gamma_2}P$) is a line perpendicular to the line through the centers of Γ_1 and Γ_2.



Proof. Let O_1 and r_1 be the center and the radius of Γ_1 and O_2 and r_2 be the center and the radius of Γ_2. Consider Cartesian coordinates where $O_1(0,0)$ and $O_2(k,0)$ with $k \neq 0$ because the circles are non-concentric.

The point $P(x,y)$ have the same power when

$$Pot_{\Gamma_1}P = Pot_{\Gamma_2}P \Leftrightarrow PO_1^2 - r_1^2 = PO_2^2 - r_2^2$$
$$\Leftrightarrow x^2 + y^2 - r_1^2 = (x-k)^2 + y^2 - r_2^2 \Leftrightarrow -r_1^2 = -2kx + k^2 - r_2^2$$
$$\Leftrightarrow 2kx = k^2 + r_1^2 - r_2^2 \Leftrightarrow x = \frac{k^2 + r_1^2 - r_2^2}{2k}$$

As x has a fixed value, we conclude that the points lie on a line perpendicular to the x axis.

Problems

13. Let ω and γ be two circles intersecting at P and Q. Let their common external tangent touch ω at A and γ at B. Prove that PQ passes through the midpoint M of AB.

14. (USAMO/2009) Given circles ω_1 and ω_1 intersecting at points X and Y, let ℓ_1 be a line through the center of ω_1 intersecting ω_2 at points P and Q and let ℓ_2 be a line through the center of ω_2 intersecting ω_1 at points R and S. Prove that if P, Q, R and S lie on a circle then the center of this circle lies on line XY.

15. (Russia/2014) A trapezoid $ABCD$ with bases AB and CD is inscribed into circle Ω. A circle ω passes through the point C and D, and intersects the segments CA and CB at $A_1 \neq C$ and $B_1 \neq D$, respectively. The points A_2 and B_2 are symmetric to A_1 and B_1 with respect to the midpoints of CA and CB, respectively. Prove that the points A, B, A_2 and B_2 are concyclic.

16. (AIME II/2019) In acute triangle ABC points P and Q are the feet of the perpendiculars from C to AB and from B to AC, respectively. Line PQ intersects the circumcircle of ΔABC in two distinct points, X and Y and. Suppose $XP = 10$, $PQ = 25$, and $QY = 15$. The value of $AB \cdot AC$ can be written in the form $m\sqrt{n}$ where m and n are positive integers, and n is not divisible by the square of any prime. Find $m + n$.

17. (Japan/2011) Let ABC be a given acute triangle and let M be the midpoint of BC. Draw the perpendicular HP from the orthocenter H of ABC to AM. Show that $AM \cdot PM = BM^2$.

18. (Iran TST/2011) In acute triangle ABC angle B is greater than angle C. Let M is midpoint of BC. Let D and E are the feet of the altitude from C and B, respectively. Let K and L are midpoint of ME and MD, respectively. If KL intersect the line through A parallel to BC in T, prove that $TA = TM$.

19. (IMO Shortlist/1995) ABC is a triangle. A circle through B and C meets the side AB again at C' and meets the side AC again at B'. Let H be the orthocenter of ABC and H' the orthocenter of $AB'C'$. Show that the lines BB', CC' and HH' are concurrent.

20. (IMO/2013) Let ABC be an acute-angled triangle with orthocentre H, and let W be a point on the side BC, lying strictly between B and C. The points M and N are the feet of the altitudes from B and C, respectively. Denote by ω_1 the circumcircle of BWN, and let X be the point on ω_1 such that WX is a diameter of ω_1. Analogously, denote by ω_2 the circumcircle of CWM, and let Y be the point on ω_2 such that WY is a diameter of ω_2. Prove that X, Y and H are collinear.

Part 4

Theorem 5. (Radical Center) Given three circles, no two concentric, the three pairwise radical axes are either concurrent or all parallel. In the last case, the three centers are collinear.

Proof. Let Γ_1, Γ_2 and Γ_3 be the three circles with centers O_1, O_2 and O_3 respectively. Let r_{12} be radical axis of Γ_1 and Γ_2 and r_{23} be the radical axis of Γ_2 and Γ_3.
If r_{12} and r_{23} are parallel, then O_1O_2 and O_2O_3 are parallel. The point O_2 is common and the three centers are collinear. The third radical axis r_{13} is perpendicular to the line through the centers and parallel to the other radical axes.
If r_{12} and r_{23} are not parallel, then they meet at point C.

$$Pot_{\Gamma_1}C = Pot_{\Gamma_2}C = Pot_{\Gamma_3}C \Rightarrow Pot_{\Gamma_1}C = Pot_{\Gamma_3}C \Rightarrow C \in r_{13}$$

Problems

21. (USAMO/1997) Let ABC be a triangle. Take points D, E, F on the perpendicular bisectors of BC, CA, AB respectively. Show that the lines through A, B, C perpendicular to EF, FD, DE respectively are concurrent (or parallel).

22. (IberoAmerican/1999) An acute triangle ΔABC is inscribed in a circle with center O. The altitudes of the triangle are AD, BE and CF. The line EF cut the circumference on P and Q.
a) Show that OA is perpendicular to PQ.
b) If M is the midpoint of BC, show that $AP^2 = 2 \cdot AD \cdot OM$.

23. (AIME I/2016) Circles ω_1 and ω_2 intersect at points X and Y. Line ℓ is tangent to ω_1 and ω_2 at A and B, respectively, with line AB closer to point X than to Y. Circle ω passes through A and B intersecting ω_1 again at $D \neq A$ and intersecting ω_2 again at $C \neq B$. The three points C, Y, D are collinear, $XC = 67$, $XY = 47$ and $XD = 37$. Find AB^2.

24. (IMO 1995) Let A, B, C and D be four distinct points on a line, in that order. The circles with diameters AC and BD intersect at X and Y. The line XY meets BC at Z. Let P be a point on the line XY other than Z. The line CP intersects the circle with diameter AC at C and M, and the line BP intersects the circle with diameter BD at B and N. Prove that the lines AM, DN, and XY are concurrent.

25. (IMO Shortlist/2009) Let ABC be a triangle. The incircle of ABC touches the sides AB and AC at the points Z and Y, respectively. Let G be the point where the lines BY and CZ meet, and let R and S be points such that the two quadrilaterals $BCYR$ and $BCSZ$ are parallelograms. Prove that $GR = GS$.

26. (IMO Shortlist/2011) Let ABC be an acute triangle with circumcircle Ω. Let B_0 be the midpoint of AC and let C_0 be the midpoint of AB. Let D be the foot of the altitude from A, and let G be the centroid of the triangle ABC. Let ω be a circle through B_0 and C_0 that is tangent to the circle Ω at a point $X \neq A$. Prove that the points D, G, and X are collinear.

Hints and Solutions

1. https://artofproblemsolving.com/wiki/index.php/2020_AMC_12B_Problems/Problem_10

2. https://artofproblemsolving.com/wiki/index.php/2019_AIME_I_Problems/Problem_6

3. https://artofproblemsolving.com/community/c6h559591p3256063

4. https://artofproblemsolving.com/wiki/index.php/1998_USAMO_Problems/Problem_2

5. https://artofproblemsolving.com/community/c6h481928p2699657

6. https://artofproblemsolving.com/wiki/index.php/2000_IMO_Problems/Problem_1

7. https://artofproblemsolving.com/wiki/index.php/2013_AMC_10A_Problems/Problem_23

8. https://artofproblemsolving.com/wiki/index.php/2019_AIME_I_Problems/Problem_15

9. The bisector AI meets the circumcircle at point M. It is well known that $IM = BM$. By the Law of Sines $IM = BM = 2R \cdot \sin\frac{A}{2}$. Let D be the projection of I on AC. On the right triangle ADI we have $\sin\frac{A}{2} = \frac{AD}{AI} \Leftrightarrow AI = \frac{r}{\sin\frac{A}{2}}$. By Power of a Point, $Pot_{(ABC)}I = OI^2 - R^2 = -AI \cdot IM = -\frac{r}{\sin\frac{A}{2}} \cdot 2R \cdot \sin\frac{A}{2} = -2Rr \Rightarrow OI^2 = R^2 - 2Rr$.

10. https://artofproblemsolving.com/community/c6h490077p2747903

11. https://artofproblemsolving.com/community/c6h516103p2902648

12. https://artofproblemsolving.com/community/c6h488825p2739321

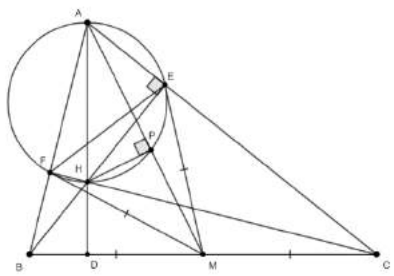
13. Let X be the intersection of PQ and AB. $XA^2 = XP \cdot XQ = XB^2 \Rightarrow XA = XB \Rightarrow X = M$.

14. https://artofproblemsolving.com/wiki/index.php/2009_USAMO_Problems/Problem_1

15. https://artofproblemsolving.com/community/c6h587995p3480814

16. https://artofproblemsolving.com/wiki/index.php/2019_AIME_II_Problems/Problem_15

17. Consider the figure. AD, BE and CF are the altitudes. P, E and F are on the circle of diameter AH, because $\angle AFH = \angle AEH = \angle APH = 90°$.



The quadrilateral $BFEC$ is cyclic and M is its circucmcenter ($\angle BFC = \angle BEC = 90°$). Using the angles of ΔAFC, we have $\angle FCE = \angle FCA = 90° - A$. By central angle, $\angle FME = 2 \cdot \angle FCE = 180° - 2A$. The triangle FME is isosceles and $\angle MFE = \angle MEF = \frac{180° - \angle FME}{2} = A$. Then $\angle MEF = \angle MFE = A = \angle EAF$, ME and MF are tangent to the circle of diameter AH and by power of a point $MP \cdot MA = ME^2 = MB^2$.

Note: this point P is known as Humpty point.

18. https://artofproblemsolving.com/community/c6h405937p2266382
19. https://artofproblemsolving.com/community/c6h29893p185022
20. https://artofproblemsolving.com/community/c6h1181533p5720174
21. https://artofproblemsolving.com/wiki/index.php/1997_USAMO_Problems/Problem_2
22. https://artofproblemsolving.com/community/c6h83883p483869
23. https://artofproblemsolving.com/wiki/index.php/2016_AIME_I_Problems/Problem_15
24. https://artofproblemsolving.com/community/c6h60435p365179
25. https://artofproblemsolving.com/community/c6h355790p1932935
26. https://artofproblemsolving.com/community/c6h488829p2739327