TREINAMENTO DE VELOCIDADE, SIMPLICIDADE E ELEGÂNCIA

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Problemas

1. Nove cientistas trabalham num projeto secreto. Por questões de segurança, os planos são guardados em um cofre protegido por muitos cadeados de modo que só é possível abri-los todos se houver pelo menos cinco cientistas presentes.

(a) Qual o número mínimo possível de chaves?

(b) Na situação do item anterior, quantas chaves cada cientista deve ter?

2. Let m and n be distinct positive integers. Represent m^6+n^6 as the sum of two perfect squares different from m^6 and n^6.

3. Prove that each positive integer not exceeding $n!$ can be represented as the sum of n or fewer distinct divisors of $n!$.

4. Four cars A, B, C, D traveled at constant speeds on the same road. A passed B and C at 8AM and 9AM, respectively, and met D at 10AM; D met B and C at 12 AM and 2PM, respectively. Determine at what time B passed C.

5. Two concentric circles S_1 and S_2 are given. Draw a line l meeting these circles consecutively in points A, B, C, D such that $AB = BC = CD$.

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6. Sejam n e k números inteiros positivos tais que $k \geq n$ e $k - n$ é um número par. São dadas $2n$ lâmpadas numeradas de 1 a $2n$, cada uma das quais pode estar *acesa* ou *apagada*. Inicialmente todas as lâmpadas estão apagadas. Uma *operação* consiste em alterar o estado de exatamente uma das lâmpadas (de acesa para apagada ou de apagada para acesa). Consideremos sequências de operações.

Seja N o número de sequências com k operações após as quais as lâmpadas de 1 a n estão todas acesas e as lâmpadas de $n+1$ a $2n$ estão todas apagadas.

Seja M o número de sequências com k operações após as quais as lâmpadas de 1 a n estão todas acesas e as lâmpadas de $n+1$ a $2n$ estão todas apagadas, e durante as quais todas as lâmpadas de $n+1$ a $2n$ permanecem sempre apagadas.

Determine a razão $\dfrac{N}{M}$.

7. Consider $a_1, a_2, \dots, a_n$ in the interval $[-2, 2]$ such that their sum is zero. Prove that

$$\left|a_1^3 + a_2^3 + \cdots + a_n^3\right| \leq 2n.$$

8. If p is a prime greater than 3 and $q = \lfloor 2p/3 \rfloor$, prove that

$$\binom{p}{1} + \binom{p}{2} + \cdots + \binom{p}{q}$$

is divisible by p^2.

9. The endpoints of a chord ST with constant length are moving along a semicircle with diameter AB. Let M be the midpoint of ST and P the foot of the perpendicular from S to AB. Prove that angle SPM is independent of the location of ST.

10. The positive integers are to be partitioned into several subsets A_1, $A_2, \ldots, A_n$ such that, for $i = 1, 2, \ldots, n$, if $x \in A_i$ then $2x \notin A_i$. What is the minimum value of n ?

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11. Let n be a positive integer. Is it possible to arrange the numbers $1, 2, \ldots, n$ in a row so that the arithmetic mean of any two of these numbers is not equal to some number between them? In other words: Is there a permutation $a_1, a_2, \ldots, a_n$ of the numbers $1, 2, \ldots, n$ such that there are no indices $i < k < j$ for which $a_k = \frac{1}{2}(a_i + a_j)$?

12. Qual é o menor valor que a expressão $\sqrt{x^2+1}+\sqrt{(y-x)^2+4}+\sqrt{(z-y)^2+1}+\sqrt{(9-z)^2+16}$ pode assumir, sendo x, y e z reais?

13. Do there exist permutations

$$a_1, a_2, \ldots, a_{50}, \quad b_1, b_2, \ldots, b_{50}, \quad c_1, c_2, \ldots, c_{50}, \quad d_1, d_2, \ldots, d_{50}$$

of the first fifty positive integers such that

$$\sum_{i=1}^{50} a_i b_i = 2 \sum_{i=1}^{50} c_i d_i?$$

14. It is well known that the divisibility tests for division by 3 and 9 do not depend on the order of the decimal digits. Prove that 3 and 9 are the only positive integers with this property. More exactly, if an integer $d > 1$ has the property that $d \mid n$ implies $d \mid n_1$, where n_1 is obtained from n through an arbitrary permutation of its digits, then $d = 3$ or $d = 9$.

15. Let $ABCD$ be a cyclic quadrilateral such that $AD + BC = AB$. Prove that the bisectors of the angles ADC and BCD meet on the line AB.

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16. Let $n > 1$ be an integer. An $n \times n \times n$ cube is composed of n^3 unit cubes. Each unit cube is painted with one color. For each $n \times n \times 1$ box consisting of n^2 unit cubes (of any of the three possible orientations), we consider the set of the colors present in that box (each color is listed only once). This way, we get $3n$ sets of colors, split into three groups according to the orientation. It happens that for every set in any group, the same set appears in both of the other groups. Determine, in terms of n, the maximal possible number of colors that are present.

17. Several counters are placed on a chessboard. At each move one of the counters goes to any vacant square it has not previously occupied. Each counter visits every square of the board and returns to its starting position at its final move. Prove that there is a moment when none of the counters occupies its initial position.

18. Let **P** be a polynomial with positive coefficients. Prove that if

$$P\left(\frac{1}{x}\right) \geq \frac{1}{P(x)}$$

holds for $x = 1$, then it holds for every $x > 0$.

19. Prove that the number of divisors of the form $4k + 1$ of each positive integer is not less than the number of its divisors of the form $4k + 3$.

20. Let ABC be a given triangle, and let k be a line parallel to BC which meets sides AC and AB in points K and L respectively; let m be a line parallel to CA meeting sides BA and BC in points M and N; let p be a line parallel to AB meeting sides CB and CA in points P and Q. Prove that the points of intersection of AB and KN, of BC and MQ, and of CA and PL, if they all exist, are collinear.