“Moving Pictures”

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Problemas

1. If a figure F is moved in the plane in such a manner that two given non-parallel lines p and q of F always pass through two given points A and B in the plane, then every other line of F either always passes through some given point in the plane, or is always tangent to some circle in the plane.

2. Now suppose that the figure F moves in the plane in such a manner that two non-parallel lines m and n of the figure are always tangent to two given circles S_1 and S_2, then every line of the figure F is either always tangent to some fixed circle, or always passes through some given point.

3. If a figure F moves in the plane in such a manner that two of its points A and B trace out lines p and q that intersect in a point O, then there exists a circle S attached to the figure F, all points of which trace out lines passing through O.

4. Construct a triangle congruent to a given triangle and whose sides

(a) pass through three given points;

(b) are tangent to three given circles.

5. (a) The hypotenuse of a right triangle slides with its endpoints on two perpendicular lines. Find the locus described by the vertex at the right angle.

(b) The longest side of an isosceles triangle with vertex angle of $120°$ slides with its endpoints on the sides of an angle of $60°$. Find the locus described by the vertex at the largest angle.

6. In the plane are given two perpendicular lines l_1 and l_2 and a circle S. Construct a right triangle ABC with a given acute angle α, whose vertices A and B lie on l_1 and l_2, and whose vertex C at the right angle lies on S.

7. Let A be one of the points of intersection of two circles S_1 and S_2. Through A we pass an arbitrary line l and a fixed line l_0, intersecting S_1 and S_2 for the second time in points M_1, M_2 and N_1, N_2; let M_1M_2P be an equilateral triangle constructed on the segment M_1M_2 and let Q be the point of intersection of the lines M_1N_1 and M_2N_2. Prove that when the line l is rotated around A,

(a) the vertex P of the triangle M_1M_2P describes a circle Σ, and the sides M_1P and M_2P turn around certain fixed points I_1 and I_2 (M_1P passes through I_1 and M_2P through I_2);

(b) Q describes a circle Γ. Find the locus described by the centers of the circles Γ corresponding to different positions for the given line l_0.

8. Let l be an arbitrary line passing through the vertex A of a triangle ABC and meeting its base BC in a point M; let O_1 and O_2 be the centers of the circles circumscribed about triangles ABM and ACM. Find the locus described by the centers of the segments O_1O_2 corresponding to all possible positions of the line l.

9. A triangle ABC and a point O are given. Through O are passed three lines l_1, l_2 and l_3, such that the angles between them are equal to the angles of the triangle (the direction of the angles is taken into account); let $\bar{A}, \bar{B}$ and $\bar{C}$ be the points of intersection of these lines with the corresponding sides of $\triangle ABC$.

(a) Prove that if O is

1) the center of the circumscribed circle;

2) the center of the inscribed circle;

3) the point of intersection of the altitudes (the orthocenter) of triangle ABC, then O is also

1) the orthocenter;

2) the center of the circumscribed circle;

3) the center of the inscribed circle of triangle $\bar{A}\bar{B}\bar{C}$.

(b) Let the point O be arbitrary and let the lines l_1, l_2 and l_3 rotate around O. Find the locus of

1° the centers of the circumscribed circles;

2° the centers of the inscribed circles;

3° the orthocenters of the triangles $\bar{A}\bar{B}\bar{C}$.

10. If the figure F moves in such a way that all positions are similar to the original position and such that some three points A, B and C of the figure describe three lines not passing through a common point, then every point of the figure describes a straight line.

11. If the figure F moves in such a way that all positions are similar to the original position and so that three lines l, m and n of F, not passing through a common point, pass at all times through three given points, then every line of F passes at all times through some constant point, and every point of F describes a circle.

12. Construct a quadrilateral $ABCD$, similar to a given one (for example, a square),

(a) whose vertices lie on four given lines;

(b) whose sides pass through four given points;

(c) whose sides BC, CD and diagonal BD pass through three given points, and whose vertex A lies on a given circle.

13. Let four lines l_1, l_2, l_3, and l_4 be given. Construct a line l with the property that the three intervals cut off on it by the four given lines form given ratios.

14. Rotate each side of triangle ABC about the midpoint of that side through a fixed angle α (in the same direction each time); let $A'B'C'$ be the new triangle formed by the rotated lines. Find the locus of points of intersection of the altitudes, of the angle bisectors, and of the medians of triangle $A'B'C'$ corresponding to different values of the angle α. Prove that the centers of the circumscribed circles of all these triangles coincide.

15. Let M, K and L be three points lying on sides AB, BC and AC of triangle ABC. Prove that

(a) the circles S_1, S_2 and S_3 circumscribed about triangles LMA, MKB, and KLC meet in a point;

(b) the triangle formed by the centers of circles S_1, S_2 and S_3 is similar to triangle ABC.

16. In a circle S let there be inscribed two directly congruent triangles ABC and $A_1B_1C_1$; let $\bar{A}, \bar{B}$, and $\bar{C}$ be the points of intersection of their corresponding sides. Prove that (a) triangle $\bar{A}\bar{B}\bar{C}$ is similar to triangles ABC and $A_1B_1C_1$; (b) the point of intersection of the altitudes of triangle $\bar{A}\bar{B}\bar{C}$ coincides with the center of the circle S.

17. Four lines are given in the plane, no three of which pass through a common point and no two of which are parallel. Prove that the points of intersection of the altitudes of the four triangles formed by these lines lie on a line.

18. Find the locus of points M such that the lengths of the tangents from M to two intersecting circles S_1 and S_2 has a given ratio.

Consider three similar figures F_1, F_2, F_3. Let O_3, O_2 and O_1 denote the centers of rotation of each successive pair of these figures. The triangle $O_1O_2O_3$ is called the triangle of similarity, and the circle circumscribed about this triangle is called the circle of similarity of the figures F_1, F_2 and F_3. In case the points O_1, O_2 and O_3 all lie on a line, or coincide, the circle of similarity degenerates to a line – the similarity axis – or to a point, the similarity center, of the figures. If F_1, F_2 and F_3 are pairwise centrally similar then the circle of similarity degenerates to either a line or a point. In the next two problems we assume that the circle of similarity of the three figures F_1, F_2 and F_3 does not degenerate to a line or a point.

19. Three similar figures F_1, F_2 and F_3 are given in the plane. Let A_1B_1, A_2B_2 and A_3B_3 be three corresponding line segments in these figures; let $D_1D_2D_3$ be the triangle whose sides are the lines A_1B_1, A_2B_2 and A_3B_3. Prove that

(a) the lines D_1O_1, D_2O_2 and D_3O_3 meet in a point U that lies on the circle of similarity of the figures F_1, F_2 and F_3;

(b) the circles circumscribed about triangles $A_1A_2D_3, A_1A_3D_2$ and $A_2A_3D_1$ meet in a point V that lies on the circle of similarity of the figures F_1, F_2 and F_3;

(c) let $D'_1D'_2D'_3$ be some triangle different from $D_1D_2D_3$, whose sides are three corresponding lines in the figures F_1, F_2 and F_3. Then the triangles $D_1D_2D_3$ and $D'_1D'_2D'_3$ are directly similar and the rotation center O of these two triangles lies on the circle of similarity of F_1, F_2 and F_3.

20. Let F_1, F_2 and F_3 be three similar figures; let l_1, l_2 and l_3 be corresponding lines in these figures, and suppose that l_1, l_2, l_3 meet in a point W. Prove that

(a) W lies on the circle of similarity of F_1, F_2 and F_3;

(b) l_1, l_2 and l_3 pass through three constant (that is, not depending on the choice of the lines l_1, l_2 and l_3) points J_1, J_2 and J_3 that lie on the circle of similarity of F_1, F_2 and F_3.

Referência

"Geometric Transformations II", I. M. Yaglom, MAA Press 1968.
Disponível em http://www.ams.org/books/nml/021/nml021-endmatter.pdf

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